

Stochastic Control in Asset Allocation: Literature Review[☆]

Ali Al-Aradi

Department of Statistical Sciences, University of Toronto, Toronto, Canada

Abstract

A high-level overview of a number of papers addressing portfolio selection problems via optimal stochastic control with a focus on habit formation and minimum consumption constraints.

Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case - Merton (1969)

This paper is one of the seminal works addressing the investment-consumption problem faced by an individual investor. Subsequent papers mostly deal with variations of the main problem considered here: to determine the optimal portfolio selection and consumption rules through time for an individual that can invest in two assets; one risky and the other riskless.

Merton begins by deriving the budget equation describing how wealth changes as a function of investment and consumption. He does so by considering the change in wealth, W , over a small time interval, $[t, t + h]$, given by:

$$\begin{aligned} W(t+h) &= \left[\sum_{i=1}^m w_i(t) \frac{X_i(t+h)}{X_i(t)} \right] \cdot [W(t) - C(t)h] \\ &= \left[\sum_{i=1}^m w_i(t) (e^{g_i(t)h} - 1) \right] \cdot [W(t) - C(t)h] - C(t)h \end{aligned} \quad (1)$$

where X_i and w_i are the price of asset i and the proportion of wealth invested in that asset, respectively, and C is the consumption per unit time. The continuously compounded rate of

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return, $g_i(t)h$, is assumed to be normally distributed with mean $(a_i - \sigma_i^2/2)h$ and variance $\frac{1}{2}\sigma_i^2h$. In the two-asset model, one asset is assumed to have a rate of return $g(t)$ with positive variance (risky asset) and the other is assumed to have to have a constant return r with zero variance (riskless asset). Taking the limit as $h \rightarrow 0$ in this two-asset case yields the stochastic differential equation for the budget equation:

$$dW = [(w(t)(a - r) + r)W(t) - C(t)] dt + w(t)\sigma W(t)dZ(t) \quad (2)$$

where $Z(t)$ is an arithmetic Brownian motion.

Next, the optimal allocation and consumption problem is formulated:

$$\max_{w,C} \mathbb{E}_0 \left[\int_0^T e^{-\rho t} U(C(t)) dt + B(W(T), T) \right] \quad (3)$$

where U is a strictly concave utility function, $B(W(T), T)$ is a bequest function, and ρ is a discount factor which can be thought of as a measure of “impatience”. To derive the optimal controls, first the value function is defined as follows:

$$I(W(t), t) = \max_{w(s), C(s)} \mathbb{E}_t \left[\int_t^T e^{-\rho t} U(C(s)) ds + B(W(T), T) \right] \quad (4)$$

Using Taylor’s theorem and the mean value theorem on the preceding equation and taking the limit as $h \rightarrow 0$, Merton arrives at the equation of optimality:

$$0 = \max_{w,C} \phi(w, C; W, t) \quad (5)$$

where $\phi = \left[e^{-\rho t} U(C(t)) + \frac{\partial I}{\partial t} + \frac{\partial I}{\partial W} ((w(t)(a - r) + r)W(t) - C(t)) + \frac{1}{2} \frac{\partial^2 I}{\partial W^2} \sigma^2 w^2(t) W^2(t) \right]$. Combining the first-order conditions for the interior maximization problem with the condition given by the equation above, we have the optimality conditions as a set of two algebraic equations and one PDE to be solved for $w^*(t)$, $C^*(t)$, and $I(W(t), t)$:

$$\begin{cases} \phi(w^*, C^*; W, t) = 0 & (6a) \\ \phi_C(w^*, C^*; W, t) = 0 & (6b) \\ \phi_w(w^*, C^*; W, t) = 0 & (6c) \\ \text{subject to } I(W(T), T) = B(W(T), T) & (6d) \end{cases}$$

The paper provides solutions for optimal consumption and allocation assuming constant relative risk aversion, i.e. $U(C) = \frac{C^\gamma}{\gamma}$, $\gamma < 1$ and $U(C) = \log(C)$, $\gamma = 0$, where $1 - \gamma$ is the

measure of risk aversion. In this case, the optimal control is given by:

$$\begin{cases} C^*(t) = [\nu / (1 + (\nu\epsilon - 1)e^{\nu(t-T)})] W(t) & \text{for } \nu \neq 0 & (7a) \\ \quad = 1/(T - t + e)W(t) & \text{for } \nu = 0 & (7b) \\ w^*(t) = \frac{a - r}{\sigma^2(1 - \gamma)} & & (7c) \end{cases}$$

where $\nu = \frac{\rho - \gamma \left(\frac{(a-r)^2}{2\sigma^2(1-\gamma)} + r \right)}{1 - \gamma}$ and the assumed bequest function is a proxy for the “no-bequest” condition: $B(W(T), T) = \epsilon^{1-\gamma} e^{-\rho T} W(T)^\gamma / \gamma$.

A few observations on these results:

- The optimal allocation to the risky asset is a constant, independent of time, wealth, horizon and the consumption decision. This allocation increases as the growth rate a increases, and decreases with volatility, risk-free rate and risk aversion.
- Since there is no utility associated with wealth past the horizon T , the optimal solution forces $W(t) \rightarrow 0$ as $t \rightarrow T$.
- In the special case, $\gamma = 0$ the consumption decision is independent of the financial parameters and depends only on level of wealth.

If we assume a constant absolute risk aversion utility function, i.e. $U(C) = -e^{-\eta C} / \eta$, where $\eta > 0$ is the measure of risk aversion, we find the following optimal decision rules:

$$\begin{cases} C^*(t) = rW(t) + \left[\frac{\rho - r + (a - r)^2 / 2\sigma^2}{\eta^r} \right] & (8a) \end{cases}$$

$$\begin{cases} w^*(t) = \frac{a - r}{\eta r \sigma^2 W(t)} & (8b) \end{cases}$$

The paper also considers the infinite horizon problem by defining a new value function, $J(W(t), t) = J(W) = e^{\rho t} I(W(t), t)$ that has no explicit dependence on time. By substituting this into the optimality equation (5), we have the new optimality condition:

$$0 = \max_{w, C} \left[U(C) - \rho J + J'(W) ((w(t)(a - r) + r)W - C) + \frac{1}{2} J''(W) \sigma^2 w^2 W^2 \right] \quad (9)$$

Subject to the boundary condition $\lim_{t \rightarrow \infty} B(W(T), T) = 0$, and assuming constant relative

risk aversion we obtain the optimal rules:

$$\left\{ \begin{aligned} C^*(t) &= \left[\frac{\rho}{1-\gamma} - \gamma \left(\frac{(a-r)^2}{2\sigma^2(1-\gamma)^2} + \frac{r}{1-\gamma} \right) \right] W(t) & (10a) \\ w^*(t) &= \frac{a-r}{\sigma^2(1-\gamma)} & (10b) \end{aligned} \right.$$

The paper also uses employs techniques from comparative statics to examine the effect of changes to the financial parameters on consumption behavior. Finally, the multiple asset case and some potential extensions to the model are given.

Optimum Consumption and Portfolio Rules in a Continuous-Time Model - Merton (1971)

This paper expands on the work of Merton's 1969 paper by approaching the stochastic control problem with more rigor (applying the HJB equation to derive optimal rules rather than using Taylor expansions) and extending the model in several ways, including:

- additions to wealth other than capital gains, e.g. (possibly stochastic) wages;
- making the "risk-free" asset defaultable;
- considering the possibility of the investor's death at a random time;
- using alternatives to geometric Brownian motion for modeling asset price behavior.

Another important contribution in this paper is the formulation of a "separation" or "mutual fund" theorem. It shows that when prices are assumed to be lognormal, one can simplify the allocation problem by considering just two assets (one risky and the other risk-free) without loss of generality.

Deriving the budget equation is done in a similar manner to the one described in the 1969 paper. The discrete-time behavior of wealth, $W(t)$, and consumption, $C(t)$, is considered over a small time interval of length h . Wealth changes as the value of the $N_i(t)$ shares held changes according to the new prices, and consumption is based on the number of shares sold over the interval:

$$W(t+h) = \sum_{i=1}^n N_i(t)P_i(t+h); \quad -C(t+h)h = \sum_{i=1}^n [N_i(t+h) - N_i(t)] P_i(t)$$

Then the limit is taken to obtain the continuous-time formulation of these processes, Ito's lemma is applied and, after identifying additions to wealth from sources other than capital gains dy , the budget equation is given (assuming one risk-free asset with $\alpha_i = r$ and $\sigma_i = 0$):

$$dW = \sum_{i=1}^m w_i(\alpha_i - r)W dt - (rW - C)dt + dy + \sum_{i=1}^m w_i W \sigma_i dz_i \quad (1)$$

where $w_i(t) = N_i(t)P_i(t)/W(t)$ and prices are assumed to follow Ito processes of the form:

$$\frac{dP_i}{P_i} = \alpha_i(P, t)dt + \sigma_i(P, t)dz_i; \quad z(t) \text{ is a multidimensional Wiener process} \quad (2)$$

The optimal allocation and consumption problem is similar to the 1969 paper:

$$\max_{w, C} \mathbb{E}_0 \left[\int_0^T U(C(t), t)dt + B(W(T), T) \right] \quad (3)$$

where the utility function U is strictly concave in C , and the bequest function B is concave in W . The value function is then defined as:

$$J(W, P, t) = \max_{w, C} \mathbb{E}_t \left[\int_t^T U(C(s), s) ds + B(W(T), T) \right] \quad (4)$$

and optimality is then defined by the HJB equation:

$$0 = \max_{w, C} \phi(w^*, C^*; W, P, t) \quad (5)$$

where $\phi(w, C; W, P, t) = U(C, t) + \mathcal{L}[J]$, and $\mathcal{L}[J]$ is the infinitesimal generator of the process J . Defining the Lagrangian $L = \phi + \lambda(1 - \sum_{i=1}^n w_i)$ to force the sum of weights to equal 1, we have the following set of first-order conditions:

$$\begin{cases} 0 = L_C(w^*, C^*) = U_C(C^*, t) - J_W \\ 0 = L_{w_k}(w^*, C^*) = -\lambda + J_W \alpha_k W + J_{WW} \sum_{j=1}^n \sigma_{kj} w_j^* W^2 + \sum_{j=1}^n J_{jW} \sigma_{kj} P_j W; \\ 0 = L_\lambda(w^*, C^*) = 1 - \sum_{i=1}^n w_i^* \end{cases}$$

Solutions for optimal consumption and allocation are then obtained by solving the system of $n + 2$ equations above for C^* , w^* and λ as functions of J_W , J_{WW} , J_{jW} , W , P , and t . These are then substituted back into the HJB equation above and the second-order PDE needs to be solved for J along with the boundary condition $J(W, P, T) = B(W, T)$. Explicit solutions are given for the case where the utility function is a member of the hyperbolic absolute risk aversion (HARA) family of the form:

$$U(C, t) = e^{-\rho t} \cdot \frac{1 - \gamma}{\gamma} \left(\frac{\beta C}{1 - \gamma} + \eta \right) \quad (7)$$

This is a rich family that includes isoelastic, exponential and quadratic utility functions. The paper goes on to show that the HARA family is the only class of concave utility functions which imply linear solutions for optimal consumption and allocation.

Incorporating certain wages (or other forms of noncapital gains income) is straightforward. If the deterministic income flow is given by $dy = Y(t)dt$, the optimality equation is:

$$0 = \max_{w, \tilde{C}} \left[V(\tilde{C}, t) + \mathcal{L}[J] \right] \quad (8)$$

where $V(\tilde{C}, t) = U(\tilde{C}(t) + Y(t), t)$ and consumption is redefined as consumption in excess of wage. Optimal decision rules are given for this case when the underlying utility function is assumed to be a member of the HARA family.

The case of stochastic wages is modeled using a Poisson process by assuming $dY = \epsilon dq$, where dq is a Poisson process with parameter λ and jumps of size 1, so that wage increases by a constant amount ϵ at random points in time. Two other ways in which Poisson processes are utilized in the investment-consumption problem include allowing the riskless asset to default by defining $dP = rdt - Pdq$, and allowing the investor to die at a random point in time, τ , defined by the Poisson process, so that the optimality criterion becomes: $\max \mathbb{E}_0 [\int_0^\tau U(C(t), t)dt + B(W(\tau), \tau)]$. In all of these cases, the budget equation and/or the HJB optimality equation need to be adjusted by taking into account the change to the infinitesimal generator of the underlying processes to incorporate the Poisson process and the mean jump size appropriately.

Finally, the paper considers alternative asset price dynamics, including one that assumes mean-reverting prices with a long-term trend towards an asymptotic “normal price”:

$$\frac{dP}{P} = \beta [\phi + vt - \log(P(t)/P(0))] dt + \sigma dz \quad (9)$$

and one where prices follow a GBM but the growth rate α follows a mean-reverting process:

$$d\alpha = \beta(\mu - \alpha)dt + \delta\sigma dz \quad (10)$$

The process of solving for optimal rules assuming these models is similar to that of the original GBM model. Solutions are presented for these two models in the two-asset case with infinite time horizon and a constant absolute risk-aversion utility function.

Portfolio Selection with Transaction Costs - Magill and Constantinides (1976)

This paper addresses the assumption of costless trading in the original Merton problem by incorporating transaction costs into the model. When transactions are introduced the investor only trades when the portfolio weights go outside a certain region around the optimal portfolio weights given by the Merton solution.

The usual assumptions are made: an investor with horizon T and known income, $y(t)$, can invest in a riskless asset with return r and m risky assets whose prices, $p_i(t)$, follow a GBM: $dp_i(t) = \alpha_i p_i(t)dt + p_i(t)dz_i(t)$, where $z(t)$ is a multidimensional Brownian motion with instantaneous covariance matrix Σ . Denoting the number of securities held by $x_i(t)$, then the value of holdings in each security is given by $s_i(t) = x_i(t)p_i(t)$. Applying Ito's lemma on this equation gives:

$$ds_i(t) = (\alpha_i s_i(t) + v_i(t))dt + s_i(t)dz_i(t) \quad (1)$$

where $v_i(t) = dx_i(t)p_i(t)$ is the value of securities purchased (resp. sold) if v_i is positive (resp. negative). Transaction costs enter the problem at this point, as they are assumed to be proportional to the value of each transaction, that is:

$$T(v_1, \dots, v_m) = \sum_{i=1}^m \chi_{v_i} v_i, \quad \text{where } \begin{cases} \chi^i & v_i > 0 \\ -\chi_i & v_i < 0 \end{cases} \quad \text{and } 0 \leq \chi^i, \chi_i < 1 \quad (2)$$

It is further assumed that transaction costs, along with income and consumption expenditure, are financed from the riskless asset, which implies:

$$ds_0(t) = \left[rs_0(t) + y(t) - c(t) - \sum_{i=1}^m (1 + \chi_{v_i})v_i(t) \right] dt \quad (3)$$

The paper proceeds to consider the dynamics s_i as the limit of equations as $\epsilon \rightarrow 0^+$ (in order to use the usual stochastic control techniques to solve the problem):

$$ds_i(t) = (\alpha_i s_i(t) + v_i(t))dt + (s_i(t) + \epsilon v_i(t))dz_i(t) \quad (4)$$

Assuming the investor wishes to maximize discounted utility gives the usual value function:

$$W(s, t) = \mathbb{E}_t \left[\int_t^T e^{-\rho(\tau-t)} u(c) d\tau \right] \quad (5)$$

which satisfies the HJB equation:

$$0 = \max_{v, c} \left[u(c) + \sum_{i=1}^m W_i(\alpha_i s_i + v_i) + W_0 \left(rs_0(t) + y - c - \sum_{i=1}^m (1 + \chi_{v_i})v_i \right) + \frac{1}{2} \sum_{i, j=1}^m W_{ij}(s_i + \epsilon v_i)(s_j + \epsilon v_j)\sigma_{ij} - \rho W + W_t \right] \quad (6)$$

If we assume further a utility function with decreasing absolute risk aversion of the form $u(c, t) = e^{-\rho t} \frac{1-\eta}{\eta} \left(\frac{\beta c}{1-\eta} + \gamma(t) \right) = e^{-\rho t} (1-\eta)^{1-\eta} \frac{\beta^\eta}{\eta} (c - \hat{c}(t))^\eta$, where $\hat{c}(t) = -\gamma(t) \frac{1-\eta}{\beta}$, we find that the optimal v^* is:

$$v^* = \frac{1}{\epsilon} \left[D \cdot \frac{\Sigma^{-1}(\alpha - rn)}{1-\eta} \left(\sum_{i=1}^m \frac{(1 + \chi_{v_i}) s_i}{1 - \epsilon(\alpha_i - r)} + Y(t) - \hat{C}(t) \right) - \bar{s} \right] \quad (7)$$

where $D = \text{diag} \left(\left\{ \frac{1-\epsilon(\alpha_i-r)}{1+\chi_{v_i}} \right\}_{i=1}^m \right)$, $Y(t) = \int_t^T y(\tau) e^{-r(\tau-t)} d\tau$, $\hat{C}(t) = \int_t^T \hat{c}(\tau) e^{-r(\tau-t)} d\tau$, $n = (1, \dots, 1)$ and $\bar{s} = (s_1, \dots, s_m)$. Note that $\xi^0 = \frac{\Sigma^{-1}(\alpha-rn)}{1-\eta}$ is the vector of optimal portfolio weights in the original Merton problem (when there are no transaction costs).

The analysis proceeds by dividing v^* by the individual's effective wealth, which is given by $w(t) = \sum_{i=0}^m s_i(t) + Y(t) + \hat{C}(t)$. Letting the portfolio weights be denoted by $\xi = (s_i/w)_{i=1}^m$ and considering the limit as $\epsilon \rightarrow 0^+$, the paper derives the signal functions which signal how and when to trade securities:

$$v_j^*(\xi, \chi_v) = [\chi_{v_j}(\xi_j^0 - 1) - 1] \xi_j + \xi_j^0 \left(1 + \sum_{\substack{i=1 \\ i \neq j}}^m \chi_{v_i} \xi_i \right), \quad j = 1, \dots, m \quad (8)$$

Notice that when transaction costs are zero, $v_j^*(\xi, 0) = \xi_j^0 - \xi_j$, so that portfolio proportions are adjusted from ξ to ξ^0 at each instant, and we return to the Merton solution.

From these signal functions, the paper proceeds to establish some important results:

When the optimal Merton solution weights are sufficiently small, i.e. $|\xi_j^0| \ll 1$:

- The investor will maintain their portfolio proportions, ξ , within a neighborhood of the Merton portfolio, ξ^0 , defined by:

$$\left[\frac{\xi_j^0}{1 + \chi^j}, \frac{\xi_j^0}{1 - \chi_j} \right] \text{ if } \xi_0^j > 0, \text{ and } \left[\frac{\xi_j^0}{1 - \chi_j}, \frac{\xi_j^0}{1 + \chi^j} \right] \text{ if } \xi_0^j < 0 \text{ for } j = 1, \dots, m.$$

- Trading only occurs when the actual allocations are outside of this region, and they are brought back to the region's boundary.
- The transaction policy for asset i is independent of the portfolio proportions and transaction rates of the other assets.
- The overall policy is independent of wealth level and remaining horizon.

- It can be shown that the investor trades each security at randomly spaced instants of time, and that the average frequency of trading in a security decreases as the transaction cost for that security increases.

More generally, it can still be claimed that there exists a region to which the investor's portfolio proportions are confined. However, the nature of this region is not defined.

With regards to consumption it is shown that consumption depends on the prevailing portfolio as well as the current level of wealth and remaining horizon.

Portfolio Selection with Transaction Costs - Davis and Norman (1990)

This paper is mostly inspired by the work of Magill and Constantinides (1976) which attempts to incorporate transaction costs in the original Merton problem. Although the authors agree with the main findings of that paper regarding the existence of a no-transaction region about the Merton portfolio, they comment that the argument used in the previous paper is heuristic and does not address the location of those boundaries, nor what the controlled process does upon reaching them.

The assumptions are mostly the same:

- The opportunity set is composed of a riskless asset with return r and a stock whose price follows a GBM with parameters α, σ^2 .
- The investor has an infinite horizon and has HARA (hyperbolic absolute risk aversion) utility, i.e. $u(c) = c^\gamma/\gamma$, ($0 < \gamma < 1$).
- Transaction costs are assumed to be proportional to transaction amount; that is, a purchase (resp. sale) of dL (resp. dU) units of stock requires a payment of $(1 + \lambda)dL$ (resp. $(1 - \mu)dU$) from the bank account.

Denoting by L_t, U_t the cumulative purchase and sale of stock in the interval $[0, t]$, and given an initial bank and stock endowment of $s_0(0) = x, s_1(0) = y$, the evolution of bank and stock holdings are given by:

$$ds_0(t) = [rs_0(t) - c(t)] dt + (1 + \lambda)dL_t + (1 - \mu)dU_t \quad (1)$$

$$ds_1(t) = \alpha s_1(t)dt + \sigma s_1(t)dz(t) + dL_t - dU_t \quad (2)$$

The investor's objective is to maximize discounted utility of consumption:

$$J_{x,y}(c, L, U) = \max_{c,L,U} \mathbb{E}_{x,y} \left[\int_0^\infty e^{-\delta t} u(c(t)) dt \right] \quad (3)$$

which allows us to define the value function as:

$$v(x, y)(c, L, U) = \sup_{c,L,U} J_{x,y}(c, L, U) \quad (4)$$

Note that this function is concave and homothetic (i.e. $v(\rho x, \rho y) = \rho^\gamma v(x, y)$).

The HJB equation is then:

$$0 = \max_{c,l,u} \left[\frac{1}{2} \sigma^2 y^2 v_{yy} + r x v_x + \alpha y v_y + \frac{1}{\gamma} c^\gamma - c v_x + \right. \\ \left. [-(1 + \lambda)v_x + v_y] l + [(1 - \mu)v_x - v_y] u - \delta v \right] \quad (5)$$

which yields the maxima:

$$c = (v_x)^{1/(\gamma-1)} \quad (6)$$

$$l = \begin{cases} \kappa & \text{if } v_y \geq (1 + \lambda)v_x \\ 0 & \text{if } v_y < (1 + \lambda)v_x \end{cases} \quad (7)$$

$$u = \begin{cases} 0 & \text{if } v_y > (1 - \mu)v_x \\ \kappa & \text{if } v_y \leq (1 - \mu)v_x \end{cases} \quad (8)$$

We can see that buying and selling either occur at the maximum rate, κ , or not at all - so we end up with three regions: “buy” (B), “sell” (S) and “no transaction” (NT). The interest now is in the shape of these boundaries. The paper uses the homothetic property of the value function (which holds in the limit as $\kappa \rightarrow \infty$), to show that these boundaries are straight lines that go through the origin in the (s_0, s_1) plane. These two lines form the wedge that contains the “Merton line” $s_1 = [\pi^*/(1 - \pi^*)]s_0$ which reflects the constant fractions of total wealth invested in the two assets given by the original Merton solution.

Putting all this information together we find that in the transaction regions, transactions occur instantaneously to return to the boundary of NT. In the NT, since optimal consumption is given by $c = (v_x)^{1/(\gamma-1)}$ and the value function satisfies the HJB equation with $l = u = 0$, we can rewrite the equation:

$$\begin{aligned} 0 &= \max_c \left[\frac{1}{2} \sigma^2 y^2 v_{yy} + (rx - c)v_x + \alpha y v_y + \frac{1}{\gamma} c^\gamma - c v_x - \delta v \right] \\ 0 &= \frac{1}{2} \sigma^2 y^2 v_{yy} + rx v_x + \alpha y v_y + \frac{1 - \gamma}{\gamma} (v_x)^{\gamma/(\gamma-1)} - \delta v \end{aligned} \quad (9)$$

Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model - Bodie et al. (1992)

This paper attempts to incorporate aspects of an individual's labor decisions into the original investment-consumption problem when there is a finite time horizon. This includes deciding on the leisure-labor split either at each point in time (flexible labor supply) or at the outset (fixed labor supply) while taking into consideration current and future wages (which may be stochastic).

The paper assumes an individual with a finite investment horizon of length T who can allocate a proportion of their financial wealth, \hat{x} , to one riskless asset with a certain return, r , and/or a risky asset whose price, $P(t)$, follows a GBM. Moreover, the individual earns wages, $w(t)$, that also follow a GBM and are assumed to be perfectly correlated with the risky asset, but less volatile:

$$\frac{dP}{P} = \alpha dt + \sigma dz; \quad \frac{dw}{w} = g dt + k\sigma dz, \quad 0 < k < 1 \quad (1)$$

Another assumption is that the individual's total wealth, $W(t)$, is the sum of financial wealth, $F(t)$, and human capital, $H(t)$. The latter represents the present value of future labor income, which is viewed as a non-traded asset and valued as if it were a traded asset. If we assume no stochasticity, the value of future wages is determined by discounting the cash flows at the risk-free rate: $H(t) = w(t)(1 - e^{-(r-g)(T-t)})/(r - g)$. In the case of stochastic wages, we use a method analogous to the general dynamic hedging argument used to price contingent claims, which gives: $H(w(t), t) = (w(t)/\mu) (1 - e^{-\mu(T-t)})$, where $\mu = r + k(\alpha - r) - g$. Furthermore it can be shown that human capital is economically equivalent to investing kH in the risky asset and $(1 - k)H$ in the risk-free asset. In other words, having risky future wages can be thought of as having an implicit investment in the risky asset.

As in the original problem, the individual must determine their level of consumption, $C(t)$, but now they must also determine the quantity of labor they will supply, $h(t)$, and by extension they amount of leisure they will consume, $L(t) = 1 - h(t)$. Thus, the labor income earned becomes $w(t)(1 - L(t))$. The dynamic budget equation is given by:

$$dW = [(x(\alpha - r) + r)W - C - wL] dt + \sigma x W dz; \quad W(0) = F(0) + H(0) \quad (2)$$

where x is the fraction of total (financial and human capital) wealth invested in the risky asset. The objective now is to maximize discounted lifetime expected utility, which depends on both consumption and leisure:

$$\max_{x, C, L} \mathbb{E}_0 \left[\int_0^T e^{-\delta t} u(C(s), L(s)) ds \right] \quad (3)$$

The following table summarizes the process of finding the optimal decision rules in the flexible and fixed labor supply cases, when wages are assumed to be nonstochastic:

	Flexible labor supply	Fixed labor supply
Value function	$J(W, w, t)$ $= \max_{x, C, L} \mathbb{E}_t \left[\int_t^T e^{-\delta t} u(C(s), L(s)) ds \right]$	$I(W, t, L)$ $= \max_{x, C} \mathbb{E}_t \left[\int_t^T e^{-\delta t} u(C(s), L) ds \right]$
HJB equation	$0 = \max_{x, C, L} \left[u(C, L) e^{-\delta t} \right.$ $\quad + J_W [(x(\alpha - r) + r)W - C - wL]$ $\quad \left. + J_t + J_w g w + \frac{1}{2} x^2 W^2 \sigma^2 J_{WW} \right]$	$0 = \max_{x, C} \left[u(C, L) e^{-\delta t} \right.$ $\quad + I_W [(x(\alpha - r) + r)W - C - wL]$ $\quad \left. + I_t + I_w g w + \frac{1}{2} x^2 W^2 \sigma^2 I_{WW} \right]$
First order conditions	$0 = u_C(C^*(t), L^*(t)) e^{-\delta t} - J_W$ $0 = u_L(C^*(t), L^*(t)) e^{-\delta t} - w J_W$ $0 = J_W(\alpha - r) + x^* \sigma^2 W J_{WW}$	$0 = u_C(C'(t), L') e^{-\delta t} - I_W$ $0 = I_W(\alpha - r) + x' \sigma^2 W I_{WW}$
Optimal decision rules	$u_L/u_C = w$ $x^* W = - (J_W/J_{WW}) [(\alpha - r)/\sigma^2]$	$L' = \operatorname{argmax} I(W(0), 0, L)$ $x' W = - (I_W/I_{WW}) [(\alpha - r)/\sigma^2]$

The difference between the two cases is that in the flexible labor supply case leisure consumption is determined continuously, while it is determined at the outset in the fixed supply setting, i.e. $L(t) = L$. Furthermore, in the fixed labor supply case the value of human capital is determined by: $H(t) = [(1 - L')w(t)] (1 - e^{-(r-g)(T-t)}) / (r - g)$.

Assuming stochastic wages changes the HJB equation to:

$$0 = \max_{x, C, L} \left[u(C, L) e^{-\delta t} + J_W [(x(\alpha - r) + r)W - C - wL] \right. \\ \left. + J_t + J_w g w + \frac{1}{2} x^2 W^2 \sigma^2 J_{WW} + J_{Ww} x w k \sigma^2 W + \frac{1}{2} k^2 w^2 \sigma^2 J_{ww} \right] \quad (4)$$

which yields the optimal portfolio allocations:

$$x^* W = - (J_W/J_{WW}) [(\alpha - r)/\sigma^2] - (J_{Ww}/J_{WW}) k w \quad (5)$$

$$x' W = - (I_W/I_{WW}) [(\alpha - r)/\sigma^2] - (I_{Ww}/I_{WW}) k w \quad (6)$$

for the flexible and fixed labor supply cases, respectively.

The final step is to translate the optimal decisions in terms of dollars invested in the risky asset, $(D^*(t), D'(t))$ and proportion of financial wealth, $(\hat{x}^*(t), \hat{x}'(t))$ invested in the risky

asset. These are given by:

$$D^*(t) = x^*W(t) - kH(w(t), t) = x^*F + (x^* - k)H(w(t), t) \quad (7)$$

$$\hat{x}^*(t) = D^*(t)/F(t) = x^* + (x^* - k)H/F \quad (8)$$

$$\hat{x}'(t) = D'(t)/F(t) = x' + (x' - k)(1 - L')H/F \quad (9)$$

By using various utility functions, the following stylized facts can be observed:

- The individual exhibits more conservative behavior as they approach retirement.
- Labor flexibility causes an individual to take greater risks in their financial investments.
- There is an inverse relationship between the level of risk of an individual's human capital, and the level of risk they chose to take on financially.
- Flexibility in labor supply is more important in the portfolio decision of individuals with a longer time horizon.

Theory of Constant Proportion Portfolio Insurance - Black and Perold (1992)

Constant proportion portfolio insurance (CPPI) is a portfolio insurance strategy that defines a cushion as the difference between wealth and a specified floor, then uses leverage to invest a constant multiple of the cushion of risky assets (up to a borrowing limit). Like other portfolio insurance strategies, CPPI is based on taking a higher (resp. lower) level of risk when the current level of wealth is higher (resp. lower), and is suitable for investors that require downside protection while maintaining their upside potential. But unlike other portfolio insurance strategies it is based on the more complex approach of option replication.

The model assumes a reserve asset, $R(t)$, and an active asset, $A(t)$, whose values follow GBM processes with means μ_A, μ_R and standard deviations σ_a, σ_R and instantaneous covariance $\sigma_{A,R}$. Transaction costs are assumed to be proportional to the size of trades, and a discretization is employed to allow for these transaction costs. Furthermore we define:

- $W(t)$, wealth at time t ;
- $F(t)$, portfolio floor - the value of a fixed number of shares of the reserve asset;
- $C(t) = W(t) - F(t)$, the cushion (wealth in excess of the floor);
- $E(t) = \min(m \cdot C(t), b \cdot W(t))$, the exposure (the investment in the active asset). The CPPI rule calls for holding the exposure at a constant multiple, m , of the cushion. When there is a borrowing limit, the exposure is capped by the maximum leverage ratio, b , where $1 \leq b < m$. It is assumed, without loss in generality, that $b = 1$;
- $S(t) = A(t)/R(t)$, the index ratio. This quantity comes about when the reserve asset is made to be the numeraire by dividing dollar quantities by $R(t)/R(0)$, which makes the value of the reserve asset and the floor constant. The index ratio increases (resp. decreases) when the active asset outperforms (resp. underperforms) the reserve asset. By using Ito's lemma, the index ratio is shown to follow a GBM with parameters $\mu_S = \mu_A - \mu_R + \sigma_R^2(1 - \beta)$ and $\sigma_S^2 = \sigma_A^2 + \sigma_R^2(1 - \beta)$ where $\beta = \sigma_{A,R}/\sigma_R^2$ is the bond beta of the active asset.

The first step is to establish the behavior of the cushion through time when the strategy is implemented. The strategy is based on rebalancing (resetting) the exposure to m times the cushion whenever E and $m \cdot C$ differ by some minimum amount. Rebalancing can be tied back to moves in the index ratio by noting that the fractional change in the index ratio is δ if and only if there is a fractional change of δ , $m\delta$ or $-(m-1)\delta/(1+m\delta)$ in E, C or E/C , respectively. When the index ratio increases E/C falls below m , which requires purchases of

the active asset and sales of the reserve - the converse is true when the index ratio falls. In other words, money is shifted towards the better-performing asset.

In particular, rebalancing occurs if there is a fractional up-move, u , or down-move, d , in the index ratio, where $(1 + u)(1 - d) = 1$. A reversal (a pair of up- and down-moves) in the index ratio affects the cushion by a factor of $\alpha = (1 + mu)(1 - md) < 1$. The term $1 - \alpha$ represents the “volatility cost” or the fraction by which the cushion falls for each reversal in the index ratio. Trading takes places after each up- and down-move in the index ratio, so the number of moves equals the number of trades. So we can write:

$$S = S_0(1 + u)^i(1 - d)^j \quad (1)$$

$$C = S_0(1 + mu)^i(1 - md)^j \quad (2)$$

where i, j are the number of up- and down-moves, respectively, and $i + j = n$. If assume that we are far from the borrowing limit and there are no transaction costs, and that the index ratio has continuous sample paths, the second equation can be rewritten as:

$$C = C_0 \cdot \alpha^{\frac{1}{2}n} \cdot (S/S_0)^\gamma \quad (3)$$

where $\alpha = \frac{1}{2} \ln[(1 + mu)/(1 - md)] / \ln(1 + u)$. This shows that the cushion depends only on the final level of the index ratio and the number of trades during the interval. Furthermore, as $u \rightarrow 0$, $C(t) \rightarrow C_0 \cdot \exp[-\frac{1}{2}(m^2 - m)\sigma^2 t] \cdot (S(t)/S_0)^m$ with probability one. It is then shown that when we consider the borrowing limit, that (a) when the limit is in effect the portfolio is in a buy-and-hold state and the cushion grows linearly (as opposed to geometrically) in S , and (b) the only effect of being temporarily at the borrowing limit is to reduce volatility cost. When transaction costs are added into the mix it can be shown that (a) sufficiently small transaction costs do not alter the basic form of the cushion, and (b) the cushion behaves as though rebalancing were occurring after a smaller up move, \hat{u} , and a larger down-move, \hat{d} - thus, transaction costs appear as an increase in volatility cost.

The paper then proceeds to show some other results of note, namely:

- As the multiple becomes large, i.e. $m \rightarrow \infty$, the payoff under CPPI approaches that of a stop-loss strategy, whether we assume discrete rebalancing or continuous, frictionless rebalancing.
- Assuming continuous rebalancing and a multiple m , the expected wealth at t when initial wealth is W (denoted by $G(W, t)$) satisfies the PDE:

$$\frac{1}{2}\sigma^2 E^2 G_{WW} + \mu E G_W - G_t = 0 \quad (4)$$

with boundary conditions $G_W(\infty, t) = \exp(\mu t)$, $G(F, t) = F$ and $G(W, 0) = W$, where $E = E(W) = \min\{W, m(W - F)\}$.

- CPPI is equivalent to investing in perpetual American options. If the parameters m, d, σ^2 are such that $d = \frac{1}{2}(m-1)\sigma^2$, then the behavior of the cushion under CPPI can be replicated by purchasing F/K perpetual American call options on the active asset, where the floor F and the strike price K satisfy:

$$K = \left[\frac{F}{(m-1)(W_0 - F)} \right]^{1/m} \cdot S_0 \cdot \frac{m-1}{m} \quad (5)$$

and the call is exercised upon reaching the borrowing limit. Additionally, if wealth level reaches a point where the borrowing limit is no longer binding, F/K' new calls must be purchased with strike price $K' = Ke^{-\delta\tau}$ where τ is the time spent at the borrowing limit.

- CPPI is optimal for a piecewise-HARA utility function with a minimum consumption constraint. That is, it is the optimal portfolio allocation decision for a utility maximization problem of the form:

$$\max \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} U(C(t)) dt \right] \quad (6)$$

subject to $c(t) \geq c_{\min}$, where $U(c) = \begin{cases} c^{1-\lambda}/(1-\lambda), & \text{for } c \geq c^* \\ U(c^*) - (c^* - c) \cdot U'(c^*), & \text{for } c < c^* \end{cases}$.

Survival and Growth with a Liability: Optimal Portfolio Strategies in Continuous Time - Browne (1997)

This paper considers the problem of an investor that can invest in a riskless asset with return r and a risky asset whose price, P_t , follows a GBM process with parameters μ, σ^2 and faces a constant liability, i.e. is forced to withdraw funds at a given rate per unit time, c . An example of such an investor would be a pension fund manager. The manager must decide on the optimal amount to be invested in the risky stock, f_t . This gives the wealth dynamics:

$$dX_t^f = \left[rX_t^f + f_t(\mu - r) - c \right] dt + f_t\sigma dW_t \quad (1)$$

Note that, unlike the original Merton problem, consumption is constant and is not a control variable. Black and Perold (1992) briefly consider a related problem, as they show that CPPI strategies are the optimal solution for a specific utility-maximizing investor with infinite horizon who is subject to a minimum consumption constraint. However, in that context consumption was a control variable (that can vary through time) alongside the allocation decision. Another difference is that in this paper the goals we are interested in are objective (in the sense of being independent of a given utility function). These goals are:

- *Survival*, when wealth is in the so-called “danger-zone.” This objective can be formulated either as maximizing the probability of reaching the complementary “safe-region” before bankruptcy, or as minimizing the discounted penalty that is paid upon bankruptcy.
- *Growth*, when wealth is in the “safe-region.” Once again, there are two formulations for this objective: minimizing the expected time to reaching a given level of wealth, or maximizing the expected discounted reward of achieving this level of wealth.

The problems considered in this paper are solved as special cases of a general optimal control problem, of the following form: let $\tau_z^f = \inf\{t > 0 : X_t^f = z\}$ denote the first hitting time of the point z by the wealth process, and $\tau^f = \min\{\tau_l^f, \tau_u^f\}$ denote the first escape time from the interval (l, u) , where $l < X_0 < u$. Then for given functions, g, h, λ , define the cost function:

$$\nu^f(x) = \mathbb{E}_x \left[\int_0^{\tau^f} g(X_t^f) e^{-\int_0^t \lambda(X_s^f) ds} dt + h(X_{\tau^f}^f) e^{-\int_0^{\tau^f} \lambda(X_s^f) ds} \right] \quad (2)$$

with value function and optimal control:

$$\nu(x) = \sup_f \nu^f(x); \quad f_\nu^*(x) = \arg \sup_f \nu^f(x) \quad (3)$$

The HJB equation in this problem is given by:

$$\sup_f \left\{ (f(\mu - r) + rc - x)v_x + \frac{1}{2}f^2\sigma^2\nu_{xx} + g - \lambda\nu \right\} = 0 \quad (4)$$

This gives the optimal control policy and the PDE that the value function satisfies:

$$f_\nu^*(x) = - \left(\frac{\mu - r}{\sigma^2} \right) \frac{\nu_x}{\nu_{xx}} \quad (5)$$

$$(rx - c)\nu_x(x) - \gamma \frac{\nu_x^2(x)}{\nu_{xx}(x)} + g(x) - \lambda(x)\nu(x) = 0 \quad (6)$$

$$\text{subject to } \nu(l) = h(l), \nu(u) = h(u)$$

where $\gamma = \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2$. The decision rule f_ν^* is verified to be optimal using the martingale optimality principle, and the two survival and growth problems are solved by repeated application of this main result.

Before solving the growth and survival problems, we need to define the safe-region and danger-zone. Denoting the bankruptcy level by a , we can see that $(c/r, \infty)$ is the safe-region, because when wealth level exceeds c/r (the safe point), the investor can invest $c/r = c \int_0^\infty e^{-rt} dt$ in a perpetual bond and pay off all future liabilities with no possibility of bankruptcy. The region $(a, c/r)$ is then the danger-zone, where there is a positive probability of bankruptcy. It is the presence of the liability that leads to a positive probability of going bankrupt and forces the investor to invest in the risky asset to avoid ruin.

The first survival objective in the danger-zone is to choose an investment policy to minimize the probability of ruin, i.e. minimize $\mathbb{P}(\tau_a^f < \infty)$ or, equivalently, maximize $\mathbb{P}(\tau_a^f = \infty)$. To this end, a related problem is solved: finding the optimal policy to maximize the probability of hitting a point b before the bankruptcy point a , where $a < X_0 < b < c/r$. By application of the main result with $l = a, u = b, \lambda = 0, g = 0, h(b) = 1, h(a) = 0$, the optimal policy and value function are given by :

$$f_V^*(x) = \frac{2r}{\mu - r} \left(\frac{c}{r} - x \right) \quad (7)$$

$$V(x : a, b) = \frac{(c - ra)^{\gamma/r+1} - (c - rx)^{\gamma/r+1}}{(c - ra)^{\gamma/r+1} - (c - rb)^{\gamma/r+1}}, \text{ for } a \leq x \leq b \quad (8)$$

It should be noted that this result holds for $b < c/r$, and that $\tau_{c/r}^* = \infty$ a.s. under f_V^* . Intuitively, as wealth approaches the boundary of the safe-region, the investor becomes more cautious and invests less in the risky asset. Therefore, an optimal policy for this survival objective does not exist, as the safe point is unattainable. This necessitates the construction of an ϵ -optimal strategy, by modifying the f_V^* policy as follows:

$$f_\delta^*(x) = \begin{cases} f_V^*(x) & \text{for } x \leq c/r - \delta \\ \kappa & \text{for } x > c/r - \delta \end{cases} \quad (9)$$

i.e. adopting the optimal policy f_V^* until wealth exceeds $c/r - \delta$ and then investing κ in the risky stock until the safe-region is reached, where κ is chosen so that the value function corresponding to this control is within ϵ of the optimal control's value function.

The next problem in the danger-zone is minimizing the discounted penalty of bankruptcy. This amounts to finding the policy that minimizes $\mathbb{E}_x \left[e^{-\lambda \tau_a^f} \right]$, where λ is the appropriate discount rate. This is solved by applying the main result with $\lambda(x) = \lambda, g = 0, h(a) = -1$ to obtain the following optimal control policy and value function:

$$f_F^*(x) = \frac{\mu - r}{\sigma^2(\eta^+ - 1)} \left(\frac{c}{r} - x \right), \quad \text{for } a < x < c/r \quad (10)$$

$$F(x) = \left(\frac{c - rx}{c - ra} \right)^{\eta^+}, \quad \text{for } a \leq x \leq c/r \quad (11)$$

where $\eta^+ = \frac{1}{2r} \left[(r + \gamma + \lambda) + \sqrt{(\gamma + \lambda - r)^2 + 4r\gamma} \right]$.

Once we are in the safe-region, we consider the objective of growth towards a target goal $b > x > c/r$. The first way to approach this problem is to minimize the expected time to the goal, $\mathbb{E}_x \left[\tau_b^f \right]$. The solution is found by an application of the main result with $g(x) = -1, \lambda = 0, h(b) = 0$ to obtain the following optimal policy and value function:

$$f_G^*(x) = \frac{\mu - r}{\sigma^2} \left(x - \frac{c}{r} \right), \quad \text{for } c/r < x < b \quad (12)$$

$$G(x) = \frac{1}{r + \gamma} \ln \left(\frac{rb - c}{rx - c} \right), \quad \text{for } c/r < x \leq b \quad (13)$$

This policy recalls the CPPI approach as it invests a constant multiple of excess wealth over the floor c/r in the risky stock, which makes the danger-zone inaccessible from above. The amount c/r is invested in the safe asset and it is used to finance the withdrawals, then the remainder of the wealth $x - c/r$ is placed into the ordinary optimal growth portfolio.

An alternative is to maximize the discounted reward of achieving the goal, which is equivalent to maximizing $\mathbb{E}_x \left[e^{-\lambda \tau_b^f} \right]$. Once again we apply the main result with $u = b, \lambda(x) = \lambda > 0, g = 0, h(b) = 1$, which gives the following optimal policy and value function:

$$f_U^*(x) = \frac{\mu - r}{\sigma^2(1 - \eta^-)} \left(x - \frac{c}{r} \right), \quad \text{for } c/r < x < b \quad (14)$$

$$U(x) = \left(\frac{rx - c}{rb - c} \right)^{\eta^-}, \quad \text{for } c/r \leq x \leq b \quad (15)$$

where $\eta^- = \frac{1}{2r} \left[(r + \gamma + \lambda) - \sqrt{(\gamma + \lambda - r)^2 + 4r\gamma} \right]$. This optimal policy also invests a constant proportion of the excess wealth above the c/r floor.

The paper also extends these optimal control problems to the multiple asset case, as well as considering a linear withdrawal rate of the form $c(x) = c + \theta x$, as opposed to a constant withdrawal rate. The optimal control in this case amount to changing the safe point using an adjusted risk-free rate: c/\tilde{r} , where $\tilde{r} = r - \theta$.

Some Notes on the Dynamics and Optimal Control of Stochastic Pension Fund Models in Continuous Time - Cairns (2000)

This paper discusses the use of stochastic modeling and optimal control techniques in the context of pension funds. It assumes a risk-free asset, R_0 , with deterministic return δ_0 and n risky assets, $\{R_i\}_{i=1}^n$, whose values follow a correlated GBM:

$$\frac{dR_i(t)}{R_i} = d\delta_i(t) = \delta_i dt + \sum_{j=1}^n \sigma_{ij} dZ_j(t); \quad \frac{dR_0(t)}{R_0} = \delta_0 dt \quad (1)$$

where $Z(t)$ is an n -dimensional Brownian motion with instantaneous covariance matrix $D = SS^T$, $S = (\sigma_{ij})_{i,j=1}^n$ and we denote by $\lambda_i = \delta_i - \delta_0$ be the risk premium for asset i . If the fund allocates p_i to asset i , the fund's instantaneous rate of return is given by:

$$\left(1 - \sum_{i=1}^n p_i\right) \delta_0 dt + \sum_{i=1}^n p_i d\delta_i(t) = (\delta_0 + p^T \lambda) dt + p^T S dZ \quad (2)$$

Additionally, the fund receives contributions, $c(t)$, and faces payable benefits, B , that are assumed to be random and driven by another (independent) Brownian motion, Z_b with expected rate B and volatility σ_b . Putting this information together, we find that the fund size $X(t)$ behaves according to the SDE:

$$dX(t) = X(t) \cdot d\delta_X(t, X(t)) + (c(t) - B)dt - \sigma_b dZ_b(t) \quad (3)$$

where $d\delta_X(t, X(t))$ is the instantaneous return on assets held by the fund. The control variables in this system are the contribution rate $c(t)$ and the asset allocation strategy $p(t)$.

Given a loss function $L(t, c, x)$, the value function of interest and the HJB equation are given by:

$$V(t, x) = \inf_{c,p} \mathbb{E}_x \left[\int_t^\infty e^{-\beta s} L(s, c(s), X(s)), X(s) ds \right] \quad (4)$$

$$0 = \inf_{c,p} \left(e^{-\beta t} L(t, c, x) + V_t + [(\delta_0 + p^T \lambda) + c - B] V_x + \frac{1}{2} V_{xx} (x^2 \cdot p^T D p + \sigma_b^2) \right) \quad (5)$$

which yields the optimal decision rules:

$$c^*(t, x) = L_c^{-1} (e^{-\beta t} V_x) \quad (6)$$

$$p^*(t, x) = - \left(\frac{V_x}{x V_{xx}} \right) D^{-1} \lambda \quad (7)$$

It is noteworthy that this optimal solution is similar to the ones found in Merton (1969) and Merton (1971), and that there is a parallel to results from modern portfolio theory whereby

the portfolio of risky assets and the cash position mimic movement along the capital market line. The paper goes on to solve the problem when a quadratic loss function of the form $L(t, c, x) = (c - c_m)^2 + 2\rho(c - c_m)(x - x_p) + (k + \rho^2)(x - x_p)^2$ is assumed (the paper also addresses solutions when power and exponential loss functions are used). The solution is based on noticing that the objective function is Markovian and time-homogeneous lead to value function of the form $V(t, x) = e^{-\beta t}F(x)$, and then proceeding to solve the resulting PDE for V and subsequently c^* and p^* . It should be noted that the optimal control strategy does not depend on the parameters of the benefit process, B and σ_b .

The paper then considers two additional variations on this problem: first, a cash constraint where a fixed proportion p_m is invested in risky assets and the rest invested in cash. This represents a cash amount that must be held (by some regulatory requirement, perhaps) to provide short-term liquidity and be used for immediate benefits payments. The only adjustment required to the solution method above is to add a Lagrangian term of the form $\gamma(e^T p - p_m)$ in the first minimization over (c, p) , where $e = (1, \dots, 1)$.

The second variation is to assume a static asset allocation strategy, i.e. $p(t, x) = p$. In this case the HJB equation is applied but the minimization is done over contribution policies, c , only. The final result is given as a function of p and a standard optimization exercise yields the optimal static allocation strategy.

Finally, the paper considers a generalization where a linear form is assumed for $c(x)$ and $xp(x)$, i.e. $c(t, x) = c_0 - c_1x$ and $p(t, x) = p_0/x + p_1$, as well as constraints and discontinuities such as: the presence of a lower and upper barrier for fund size, limits on contribution rates and restrictions on short-sale of assets.

Optimal Dynamic Strategies with Risk Limits - Cuoco et al. (2008)

The focus in this paper is on determining optimal portfolio allocation subject to constraints expressed in terms of risk limits in the form of value-at-risk (VaR) and tail conditional expectation (TCE). The opportunity set consists of a riskless asset with return r and n risky assets that follow an n -dimensional GBM with drift vector $r\mathbb{1} + \mu$ and diffusion matrix σ . Given a portfolio-weight process, π , the portfolio value process, W^π satisfies:

$$\begin{aligned} W_t^\pi &= W_0 + \int_0^t W_s^\pi \left(r + \pi_s^\top \mu \right) ds + \int_0^t W_s^\pi \pi_s^\top \sigma dw_s \\ &= W_0 \exp \left[\int_0^t \left(r + \pi_s^\top \mu - \frac{1}{2} \left| \pi_s^\top \sigma \right|^2 \right) ds + \int_0^t \pi_s^\top \sigma dw_s \right] \end{aligned} \quad (1)$$

It is assumed that the risk of the trading portfolio is continuously reevaluated, and that the distribution of portfolio value at the chosen horizon is computed assuming that the current portfolio composition is kept unchanged over the horizon. That is, given a portfolio π_t and the associated value W_t^π , the future value at time $t + \tau$, $W_{t+\tau}(W_t^\pi, \pi_t)$, assumes that π_t is held constant between t and $t + \tau$. This reflects the current practice in financial institutions where the future portfolio choices of traders are unknown over the VaR horizon. This gives the related definition for VaR and TCE:

$$VaR_t^{\alpha, \pi} = \inf \{ L \geq 0 : \mathbb{P}(W_t^\pi - W_{t+\tau}(W_t^\pi, \pi_t) \geq L | \mathcal{F}_t) < \alpha \} = (Q_t^{\alpha, \pi})^-, \quad (2)$$

$$\text{where } Q_t^{\alpha, \pi} = \sup \{ L \in \mathbb{R} : \mathbb{P}(W_{t+\tau}(W_t^\pi, \pi_t) - W_t^\pi \leq L | \mathcal{F}_t) < \alpha \},$$

$$x^- = \max[0, -x]$$

$$TCE_t^{\alpha, \pi} = \left(\frac{\mathbb{E} \left[W_t^\pi - W_{t+\tau}(W_t^\pi, \pi_t) \mathbb{I}_{\{W_t^\pi - W_{t+\tau}(W_t^\pi, \pi_t) \geq -Q_t^{\alpha, \pi}\}} \right]}{\alpha} \right)^+ \quad (3)$$

$$\text{where } x^+ = \max[0, x]$$

Assuming lognormally-distributed returns, VaR and TCE can be computed explicitly:

$$VaR_t^{\alpha, \pi} = W_t^\pi \left[1 - \exp \left(\left(r + \pi_t^\top \mu - \frac{1}{2} \left| \pi_t^\top \sigma \right|^2 \right) \tau + \Phi^{-1}(\alpha) \left| \pi_t^\top \sigma \right| \sqrt{\tau} \right) \right]^+ \quad (4)$$

$$TCE_t^{\alpha, \pi} = W_t^\pi \left[1 - \exp \left(\left(r + \pi_t^\top \mu \right) \tau \right) \frac{\Phi \left(\Phi^{-1}(\alpha) - \left| \pi_t^\top \sigma \right| \sqrt{\tau} \right)}{\alpha} \right]^+ \quad (5)$$

The stochastic control problem is to maximize the utility of terminal wealth subject to the constraint that portfolio VaR does not exceed a prespecified level $\overline{VaR}(W_t^\pi, t) \geq 0$, i.e.

$$\begin{aligned} &\max_{\pi} \mathbb{E} [u(W_T^\pi)] \\ \text{s.t. } &\log \left(1 - \frac{\overline{VaR}(W_t^\pi, t)}{W_t^\pi} \right)^+ - \left(r + \pi_t^\top \mu - \frac{1}{2} \left| \pi_t^\top \sigma \right|^2 \right) \tau - \Phi^{-1}(\alpha) \left| \pi_t^\top \sigma \right| \sqrt{\tau} \leq 0 \end{aligned} \quad (6)$$

where the constraint is equivalent to the constraint $VaR_t^{\alpha, \pi} \leq \overline{VaR}(W_t^\pi, t)$. Letting the $V(W, t)$ denote the value function, we can write the HJB equation as:

$$0 = \max_{\pi} \left[\frac{1}{2} V_{WW} \left| W \pi^\top \sigma \right|^2 + V_W W (r + \pi^\top \mu) + V_t - \psi \left(\log \left(1 - \frac{\overline{VaR}}{W} \right)^+ - \left(r + \pi^\top \mu - \frac{1}{2} \left| \pi^\top \sigma \right|^2 \right) \tau - \Phi^{-1}(\alpha) \left| \pi_t^\top \sigma \right| \sqrt{\tau} \right) \right] \quad (7)$$

where ψ is a Lagrangian multiplier. Then the optimal portfolio is given by:

$$\pi^*(W, t) = \varphi(W, t) \left(\sigma \sigma^\top \right)^{-1} \mu \quad (8)$$

$$\text{where } \varphi(W, t) = \min \left[-\frac{V_W(W, t)}{W V_{WW}(W, t)}, \varphi_\alpha^+(W, t) \right],$$

$$\text{and } \varphi_\alpha^+(W, t) = \frac{|\kappa| \sqrt{\tau} + \Phi^{-1}(\alpha) + \sqrt{(|\kappa| \sqrt{\tau} + \Phi^{-1}(\alpha))^2 - 2 \left(\log \left(1 - \frac{\overline{VaR}(W, t)}{W} \right)^+ - r\tau \right)}}{|\kappa| \sqrt{\tau}}$$

Note that the optimal portfolio for a VaR-constrained investor is a combination of the riskless asset and the growth-optimal portfolio $(\sigma \sigma^\top)^{-1} \mu$, which is the portfolio that maximizes the expected continuously compounded rate of return. The paper then gives explicit solutions assuming CRRA utility and for various assumptions on the VaR constraint (constant VaR limit: $\overline{VaR}(W, t) = \beta$; proportional VaR limit: $\overline{VaR}(W, t) = \beta W$; VaR limit as a fixed proportion of initial portfolio value plus any running gain: $\overline{VaR}(W, t) = (W - (1 - \beta)W_0)^+$).

The second problem is that of finding the optimal portfolio for a TCE-constrained investor:

$$\begin{aligned} & \max_{\pi} \mathbb{E} [u(W_T^\pi)] \\ & \text{s.t. } TCE_t^{\hat{\alpha}, \pi} \leq \overline{TCE}(W_t^\pi, t) \end{aligned} \quad (9)$$

Rather than solving this problem in isolation, a solution is found by mapping the dynamic TCE risk limit to an equivalent VaR limit.

Optimal Investment Decisions When Time-Horizon is Uncertain - Blanchet-Scalliet et al. (2008)

This paper extends the investment decision results involving an investor with an exponentially distributed time until death discussed in Merton (1971) to more general modeling of exit time. This is to account for other factors that can affect exit time such as: market behavior, changes in opportunity set, exogenous shocks to investor wealth and/or consumption. The investor's time-horizon, τ , is defined as the maximum length of time for which an investor gives any weight their utility function. The presence of this uncertain time-horizon introduces new uncertainty to the economy that is not necessarily independent of the market risk. To separate the two sources of risk we condition on \mathcal{F}_t to isolate the pure timing uncertainty. For this, define $F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$. Note that τ is not assumed to be a stopping time and that observing asset prices up to date t does not imply knowledge of whether or not τ has occurred by time t , i.e. there are times $t \geq 0$ such that the event $\{t < \tau\}$ does not belong to \mathcal{F}_t .

Other than this conditional distribution function of timing uncertainty, many of the assumptions are the same. The market consists of a riskless asset with return r and n risky assets that follow a multidimensional GBM with drift vector $\mu = (\mu^i)_{i=1}^n$ and instantaneous covariance matrix $\sigma = (\sigma^{i,j})_{i,j=1}^n$. Given a portfolio strategy π , the wealth process evolves according to the SDE:

$$dX_s^{t,\pi,x} = X_s^{t,\pi,x} r_s ds + \pi_s [(\mu_s - \mathbb{1}r_s) ds + \sigma dW_s] \quad (1)$$

and the investor's objective is to maximize the expected utility of terminal wealth, which can be expressed using the conditional distribution of the random time of exit:

$$\begin{aligned} V(x) &= \sup_{\pi} \mathbb{E} [U(X_{\tau \wedge T}^{\pi,x})] \\ &= \sup_{\pi} \mathbb{E} \left[\int_0^{\infty} U(X_{t \wedge T}^{\pi,x}) dF_t \right] = \sup_{\pi} \mathbb{E} \left[\int_0^{\infty} U(X_u^{\pi,x}) dF_u + U(X_T^{\pi,x})(1 - F_T) \right] \end{aligned} \quad (2)$$

Assuming F_t is a deterministic process with derivative f , the value function is given by:

$$V(t, x) = \max \mathbb{E} \left[\int_t^T U(X_s^{t,\pi,x}) f(s) ds + U(X_T^{t,\pi,x})(1 - F(T)) \right] \quad (3)$$

The associated HJB equation and optimal portfolio strategy are then given by:

$$\begin{aligned} 0 &= f(t)U(x) + \left(V_t + \sup_{\pi} A(t, x, \pi) \right) \quad \text{subject to } V(T, x) = U(x)(1 - F(T)) \\ \pi_t^* &= -\sigma_t^{-1} \theta_t \frac{V_x}{V_{xx}}, \quad \text{where } \theta_t = \sigma_t^{-1} (\mu_t - \mathbb{1}r_t) \end{aligned} \quad (4)$$

where $A(t, x, \pi) = [xr_t + \pi(\mu_t - r_t)]V_x + \frac{1}{2}\pi^2\sigma_t^2V_{xx}$.

Explicit solutions are derived for the case of logarithmic ($U(x) = \ln x$) and power ($U(x) = x^\alpha/\alpha$) utility functions. The solutions coincide with those of the fixed time-horizon case, regardless of the distribution F of time-horizon. This extends the result in Merton (1971) to general distribution of time-horizon. The intuition for this result is based on the fact that CRRA utility function and deterministic coefficients do not depend on the time-horizon.

The next result is for the same problem(s), but with an infinite time span. Here, the value function and HJB equation are almost identical to the previous case except for different limits of integration and boundary condition:

$$V(t, x) = \sup_{\pi} \mathbb{E} \left[\int_t^{\infty} f(u)U(X_u^{t,\pi,x})du \right] \quad (5)$$

$$0 = f(t)U(x) + V_t + \sup_{\pi} A(t, x, \pi) \quad \text{subject to} \quad \lim_{t \rightarrow \infty} V(t, x) = 0 \quad (6)$$

The final problem considered is when the density process is stochastically time-varying. This allows for a possible correlation between the instantaneous probability of exit and risky asset returns. For this, another Brownian motion is introduced, W^f , which is correlated with the Brownian motions driving asset returns with correlation $\sigma_{s,f} = (\sigma_{if})_{1 \leq i \leq n}$. The density function, f , is assumed to evolve according to the SDE:

$$\begin{aligned} df_s &= f_s(a(s)ds + b(s)dW_s^f) & f_0 &= y, 0 \leq s \leq T \\ \Rightarrow f(s) &= y \exp\left(\int_0^s a(u)du\right) \xi_s \end{aligned} \quad (7)$$

$$\text{where } \xi_s = \exp\left(\int_0^s b(u)dW_u^f - 1/2 \int_0^s b^2(u)du\right)$$

Note that this formulation does not prevent F from taking on values greater than 1. However, this problem maybe neglected if the parameters are such that the probability of getting forbidden values remains sufficiently small (similar to the Vasicek model that allows for negative interest rates). Assuming constant parameters and CRRA preferences, the value function (which now depends on t, x, y) is given by:

$$V(0, x, y) = \sup_{\pi} \mathbb{E} \left[\int_0^{\infty} U(X_{u \wedge T}^x) f_u du \right] = \sup_{\pi} \mathbb{E}_{\mathbb{Q}} \left[y \int_0^{\infty} \exp(A(u))U(X_{u \wedge T}^x)du \right] \quad (8)$$

where $d\mathbb{Q} = \xi_T d\mathbb{P}$ (note that is not an equivalent martingale measure). Solving for the optimal portfolio in this setting requires us to notice:

$$V(0, x, y) = y \left[\int_0^{\infty} \exp(A(u))du \right] \cdot \sup_{\pi} \mathbb{E}_{\mathbb{Q}} \left[\int_0^{\infty} \varphi(u)U(X_{u \wedge T}^x)du \right] \quad (9)$$

where $\varphi(u) = (1/\int_0^\infty \exp(A(s))ds) \cdot \exp(A(u))$. Then, we can write:

$$V(0, x, y) = y \left[\int_0^\infty \exp(A(u))du \right] \cdot V^1(0, x) \quad (10)$$

where V^1 given above is the solution associated with the density φ and wealth process $dX_s^{t,\pi,x} = X_s^{t,\pi,x} r_s ds + \pi_s [(\mu_s + \sigma_s \sigma_{S,f} b(s) - r_s) ds + \sigma_s dW_s^{\mathbb{Q}}]$, where $W_t^{\mathbb{Q}} = W_t + \int_0^t \sigma_{S,f} b(s) ds$. This gives explicit solutions for the optimal portfolio strategy for logarithmic utility:

$$V(t, x, y) = -e^{at} \frac{y}{a} \left(\frac{C}{a} (e^{a(T-t)} - 1) + \ln x \right) \quad (11)$$

$$\pi_t^* = \left[\frac{\mu - r + \sigma b \sigma_{S,f}}{\sigma^2} \right] X_t^* \quad (12)$$

where $C = r + \frac{1}{2} ((\mu - r + \sigma b \sigma_{S,f})^2 / \sigma^2)$.

And for the power utility case:

$$V(t, x, y) = -\frac{y x^\alpha}{a \alpha} e^{-Ct} \left[\left(\frac{C}{C+a} \right) e^{(a+C)T} + \left(\frac{a}{C+a} \right) e^{(a+C)t} \right] \quad (13)$$

$$\pi_t^* = - \left[\frac{\mu - r + \sigma b \sigma_{S,f}}{\sigma^2} \right] \frac{X_t^*}{\alpha - 1} \quad (14)$$

where $C = r - \frac{1}{2} \left(\frac{(\mu - r + \sigma b \sigma_{S,f})^2}{\sigma^2(\alpha - 1)} \right)$

Minimizing the Probability of Lifetime Ruin Under Borrowing Constraints - Bayraktar and Young (2007a)

This paper deals with the concept of lifetime ruin, defined as an individual outliving their wealth, and how the individual should invest in order to minimize the probability of this event occurring. The allocation is done assuming constraints on borrowing - first, the case of no borrowing is considered followed by the case of limited borrowing. The borrowing constraints are introduced because the investor exhibits some unrealistic behavior in the unconstrained case. In particular, the individual borrows more money at lower levels of wealth to invest as much as possible in the risky asset in an effort to avoid ruin, which is not a reasonable strategy.

The setting for the problem is an investor whose consumption is assumed to be constant ($c(w) = c$) or proportional to wealth ($c(w) = pw$). The investor can invest in a two-asset market (riskless asset with return r and risky asset with price process following a GBM with parameters μ, σ). When the investor allocates $\pi_{0,t}$ to the risky asset (the 0 subscript represents the no borrowing constraint), their wealth follows the SDE:

$$dW_t = [rW_t + (\mu - r)\pi_{0,t} - c(W_t)] dt + \sigma\pi_{0,t}dB_t \quad (1)$$

Note that the infinitesimal generator for a function h of this process and some control α is given by: $\mathcal{L}^\alpha h(w) = [rw + (\mu - r)\alpha - c(w)] h'(w) + \frac{1}{2}\sigma^2\alpha^2 h''(w) - \lambda h(w)$. Additionally, the investor is assumed to live up to an exponentially distributed time of death, τ_d .

Similar to Browne (1997), the wealth level c/r represents a “safe-point” at which the investor can place all their wealth in the riskless asset and the probability of lifetime ruin is zero assuming constant consumption. If we denote the first time that wealth a given level z by τ_z , and set $\tau = \tau_d \wedge \tau_{c/r}$, then the minimum probability of lifetime ruin is defined as $\psi_0(w) = \inf_{\pi_0} \mathbb{P}(\tau_0 < \tau | W_0 = w)$. The HJB equation characterizing the function ψ_0 and the optimal control is:

$$\begin{aligned} \mathcal{L}^{\pi_0^*(w)} \psi_0(w) &= 0 \text{ for } w \in (0, c/r) \\ \text{subject to: } \psi_0(0) &= 1, \psi_0(w) = 0 \text{ for } w \geq c/r \end{aligned} \quad (2)$$

Note that in the proportional consumption case there is no safe-point and ruin occurs when wealth reaches some level $w_0 > 0$ (and τ_0 is then the first hitting time for that level instead of 0). Thus the boundary conditions become $\psi_0(w_0) = 1$ and $\lim_{w \rightarrow \infty} \psi_0(w) = 0$.

To solve for the required probability function and the optimal portfolio in the constrained case (with either constant or proportional consumption), the authors start with the solutions to the corresponding problems in the unconstrained case, $\psi(w)$ and $\pi^*(w)$. Then they suppose that constrained solutions are truncated versions of the unconstrained solutions. That is, they hypothesize that there is a certain wealth level w_l , where the riskless asset allocation is 0 below and positive above. The two intervals $(w_l, c/r]$ and $[0, w_l]$ are considered separately under the hypotheses $\pi_0^* \in [0, w)$ (some riskless asset allocation) and $\pi_0^* = w$ (no riskless allocation), respectively. It is verified that the optimal allocation coincides with the unconstrained case in the first interval and then is capped at w when wealth is in the second interval to reflect the borrowing constraint. The minimum probability function on the first interval is simply a multiple of its counterpart in the unconstrained case, solved for using the Legendre transform: $\tilde{h}(v) = \min_w [h_0(w) + wv]$. In the second interval the probability is found by solving the HJB equation assuming $\pi_0^* = w$ and boundary conditions $h_0(0) = 1$ and $\frac{h_0(w_l)}{h_0'(w_l)} = -\frac{1}{d} (\frac{c}{r} - w_l)$. The critical point w_l is derived assuming the continuity of the allocation decision with respect to wealth level.

The paper then turns to the case where the individual is allowed to borrow money at a rate $b > r$. The individual allocates $\pi_{b,t}$ to the risky asset with $(W_t - \pi_{b,t})_+$ in the riskless asset and $(\pi_{b,t} - W_t)_+$ borrowed. This gives the wealth equation:

$$dW_t = [r(W_t - \pi_{b,t})_+ - b(\pi_{b,t} - W_t)_+ + \mu\pi_{b,t} - c(W_t)] dt + \sigma\pi_{b,t}dB_t \quad (3)$$

and the infinitesimal generator is $D^\alpha h(w) = [bw + (\mu - b)\alpha - c(w)] h'(w) + \frac{1}{2}\sigma^2\alpha^2 h''(w) - \lambda h(w)$. The HJB equation for the optimal control and minimum probability function is of the same form as the previous case:

$$\begin{aligned} \mathcal{D}^{\pi_b^*(w)} \psi_b(w) &= 0 \text{ for } w \in (0, c/r) \\ \text{subject to: } \psi_b(0) &= 1, \psi_b(w) = 0 \text{ for } w \geq c/r \end{aligned} \quad (4)$$

The next steps follow a process similar to the no borrowing problem, but with hypotheses in three regions: borrowing in the region $[0, w_b)$, no riskless allocation in $[w_b, w_l]$ and positive allocation to the riskless asset in $(w_l, c/r]$. The solutions follow the same general process as the previous case (reduced HJB equation in the middle region and using the Legendre transform in the other two). The optimal allocation coincides with the solution in the no borrowing case in the region $[w_b, c/r]$ (truncated version of the unconstrained solution), and the minimum probability function is another multiple of the unconstrained case on $[w_l, c/r]$ and the solution of reduced HJB equation elsewhere.

Correspondence between Lifetime Minimum Wealth and Utility of Consumption - Bayraktar and Young (2007b)

This paper considers two problems: the investment-consumption problem in the classical Merton problem where the investor decides on allocation and consumption strategies so that expected discounted utility is maximized, and an investor facing an exogenous consumption level that minimizes some function of lifetime minimum wealth. The aim of the paper is to establish when the two investors behave similarly, which turns out to be when the utility function is HARA (hyperbolic absolute risk aversion).

The model assumed is as follows: an investor that lives up to a exponentially distributed time of death τ_d can invest in a two-asset market (riskless asset with return r and risky asset with price process following a GBM with parameters μ, σ). When the investor allocates π_t to the risky asset their wealth follows the SDE:

$$dW_t = [rW_t + (\mu - r)\pi_t - c(W_t)] dt + \sigma\pi_t dB_t \quad (1)$$

A related process is the minimum wealth process:

$$M_t = \left[\inf_{0 \leq s \leq t} W_s, \tilde{M}_0 \right] \quad (2)$$

Thus, lifetime minimum wealth is given by M_{τ_d} and the value function for an investor that minimizes some function of this random variable is given by:

$$V^f(w, m) = \inf_{\pi} \mathbb{E} [f(M_{\tau_d})] \quad (3)$$

If $f(m) = \mathbb{1}_{\{m \leq b\}}$ the value function represents the minimum probability of lifetime ruin with ruin level b . The paper works with indicator functions of this form for the remainder of the paper and note that the results can be extended to any nonincreasing, nonnegative function using the monotone convergence theorem (since these functions can be expressed as the pointwise limit of a sequence of linear combinations involving indicator functions). For indicator functions of this form the value function is given by:

$$V(w, m; b) = \begin{cases} 1, & \text{if } m \leq b; \\ \psi(w; b) & \text{if } m > b \end{cases} \quad (4)$$

where $\psi(w; b)$ is the probability that wealth reaches the ruin level b before the individual dies.

A verification lemma is derived for ψ based on the HJB equation:

$$\lambda\psi(w; b) = (rw - c(w))\psi'(w; b) + \min_{\pi} \left[(\mu - r)\pi\psi'(w; b) + \frac{1}{2}\sigma^2\pi^2\psi''(w; b) \right] \quad (5)$$

with the boundary conditions $\psi(b; b) = 1, \psi(w^s; b) = 0$, where $w^s = \inf\{w : c(w) = rw, w \geq 0\}$ represents the safe-point at which the investor can place all their wealth in the riskless asset and avoid lifetime ruin (similar to Browne (1997)). The solution can be written as $\psi(w; b) = h(w)/h(b)$ where h solves

$$\lambda h(w) = (rw - c(w))h'(w) + \min_{\pi} \left[(\mu - r)\pi h'(w) + \frac{1}{2}\sigma^2\pi^2 h''(w) \right] \quad (6)$$

with boundary conditions $h(0) = 1, h(w) = 0$ for $w \geq w^s$. And the optimal portfolio is given as:

$$\pi^*(w; b) = -\frac{\mu - r}{\sigma^2} \frac{h'(w)}{h''(w)} \quad (7)$$

Using the monotone convergence theorem it can be shown that this portfolio is the optimal portfolio for the general minimization problem stated in the outset of this section, that is: $V^f(w, m) = \mathbb{E} [f(M_{\tau_d}^{\pi^*})]$.

Next the paper equates the optimal investment strategies for the problem discussed above and for the classical Merton problem. Recall that in the latter, the investor maximizes the expectation of discounted utility, so that the value function and HJB equation are defined by:

$$V^u(w) = \sup_c \mathbb{E} \left[\int_0^{\tau_d} e^{-\rho s} u(c_s) ds \right] \quad (8)$$

$$(\rho + \lambda)V^u = rw \cdot (V^u)' + \max_{c \geq 0} [u(c) - c \cdot (V^u)'] + \max_{\pi} \left[(\mu - r)\pi \cdot (V^u)' + \frac{1}{2}\sigma^2\pi^2 (V^u)''(w) \right] \quad (9)$$

with the optimal investment and consumption strategy satisfying:

$$\pi^u(w) = -\frac{\mu - r}{\sigma} \frac{(V^u)'(w)}{(V^u)''(w)} \quad (10)$$

$$u'(c^u(w)) = (V^u)'(w) \quad (11)$$

It can be shown that when the optimal allocation strategies are equated and the minimizer of lifetime ruin is assumed to follow the consumption dictated by the maximizer of utility, this consumption strategy is a linear function of wealth with slope equal to the personal discount rate ρ . In other words, the only case where the investors adopt the same strategy is when consumption is a linear function of wealth and rates of consumption coincide.

Based on this fact, the next step taken is to solve for minimum probability of lifetime ruin under the assumption of a piecewise linear consumption function. This calls for solving the HJB equation involving the function unknown h given above, which in turn requires the use of the Legendre transform of this function: $\tilde{h}(y) = \min_{w < w^s} [h_0(w) + wy]$. This then enables us to

solve for the optimal portfolio (which depends on the first and second derivatives of h).

Finally, the main result of this paper is given: in each of the tree cases $\rho > r$, $\rho = r$, $\rho < r$, when the optimal investment and consumption strategies in the two optimization problems are equated, the implied utility function exhibits hyperbolic absolute risk aversion (HARA).

Portfolio Choice with Stochastic Investment Opportunities: A User's Guide - Liu and Muhle-Karbe (2013)

This paper presents an overview of works that deal with the portfolio choice problem for an investor that looks to maximize utility of terminal wealth when the model for asset price dynamics departs from the usual constant-parameter GBM assumption. Preferences are described by a power utility function with constant relative risk aversion, i.e. $U(x) = x^{1-\gamma}/(1-\gamma)$, $\gamma > 0, \gamma \neq 1$. Also, the market consists of a riskfree asset with return r and a risky asset which evolves according to the SDE:

$$\begin{aligned}\frac{dS_t}{S_t} &= (\mu(Y_t) + r) dt + \sigma(Y_t)dW_t \\ dY_t &= b(Y_t)dt + a(Y_t)dW_t^Y\end{aligned}\tag{1}$$

Y_t is a state variable which determines the instantaneous excess return and volatility by deterministic functions μ, σ . W_t and W_t^Y are assumed to have a constant correlation ρ . This specification allows for more flexibility in modeling asset dynamics.

The three main models considered in this paper are:

1. *Black-Scholes model, or constant opportunity set (COS)*: expected excess return and volatility are constant, $\mu(y) = \mu, \sigma(y) = \sigma$, and there is no state variable.
2. *Heston-type model, or stochastic volatility (SV)*: expected excess return is assumed to be a constant risk premium per unit variance and volatility is assumed to be time-varying and mean-reverting:

$$\begin{aligned}\frac{dS_t}{S_t} &= (\mu_S Y_t + r) dt + \sqrt{Y_t}dW_t \\ dY_t &= \lambda_Y(\bar{Y} - Y_t)dt + \sigma_Y \sqrt{Y_t}dW_t^Y\end{aligned}\tag{2}$$

3. *Kim and Omberg model, or predictable returns (PR)*: expected excess return is assumed to follow a mean-reverting process and volatility is assumed to be constant:

$$\begin{aligned}\frac{dS_t}{S_t} &= (Y_t + r) dt + \sigma dW_t \\ dY_t &= \lambda_Y(\bar{Y} - Y_t)dt + \sigma_Y dW_t^Y\end{aligned}\tag{3}$$

To solve for the optimal investment strategies under these models, the wealth process, value

function, HJB equation and optimal allocation are given by:

$$\frac{dX_t^\pi}{X_t^\pi} = (r + \pi_t \mu(Y_t))dt + \pi_t \sigma(Y_t) dW_t \quad (4)$$

$$V(t, x, y) = \sup_{\pi} \mathbb{E}_t [U(X_T^\pi)]; \quad V(T, x, y) = U(x) \quad (5)$$

$$0 = V_t + \sup_{\pi} \left\{ V_x x r + V_x x \pi \mu + \frac{1}{2} V_{xx} x^2 \pi^2 \sigma^2 + V_{xy} x \pi \rho \sigma a + \frac{1}{2} V_{yy} a^2 \right\} \quad (6)$$

$$\Rightarrow \hat{\pi}(t, x, y) = -\frac{V_x(t, x, y)}{x V_{xx}(t, x, y)} \frac{\mu(y)}{\sigma(y)^2} - \frac{V_{xy}(t, x, y)}{x V_{xx}(t, x, y)} \frac{\rho a(y)}{\sigma(y)} \quad (7)$$

Using the optimal investment strategy, the HJB equation can be rewritten as:

$$V_t = \frac{1}{2} \frac{V_x^2}{V_{xx}} \frac{\mu}{\sigma} + \frac{V_x V_{xy} \mu \rho a}{V_{xx} \sigma} + \frac{1}{2} \frac{V_{xy}^2}{V_{xx}} \rho^2 a^2 - V_x x r - V_y b - \frac{1}{2} V_{yy} a^2 \quad (8)$$

Using the homothetic property of the power utility function $U(x) = x^{1-\gamma} U(1)$ allows us to write the reduced value function $v(t, y) = (1 - \gamma)V(t, 1, y)$. Then the optimal strategy and HJB equation can be rewritten as:

$$\hat{\pi}_t = \frac{\mu}{\gamma \sigma^2} + \frac{\rho a}{\gamma \sigma^2} \frac{v_y}{v} \quad (9)$$

$$v_t = \frac{\gamma - 1}{\gamma} \left(\left(\frac{\mu^2}{2\sigma^2} + \gamma r \right) v + \frac{\mu \rho a}{\sigma} v_y + \frac{\rho^2 a^2}{2} \frac{v_y^2}{v} \right) - v_y b - \frac{1}{2} v_{yy} a^2 \quad (10)$$

and the corresponding terminal condition becomes: $v(T, y) = 1$. Solutions for the three models involve applying the equations above on each model and solving the resulting ODEs.

The next section in the paper considers long-run asymptotics by replacing the maximization of terminal utility with the maximization of long-run growth rate: $\liminf_{T \rightarrow \infty} \frac{1}{T} \log U^{-1}(\mathbb{E}[U(X_T^\pi)])$. Solutions for this problem involve developing an ansatz for the value function based on the limiting form of the ODEs in the original problem of terminal wealth maximization as the time to horizon $T - t$ tends to infinity. This gives sets of equations that can be solved to obtain the strategies that maximize long-run growth rate.

The final section in the paper develops verification theorems for the solutions using tools from convex duality.

Portfolio Choice with Illiquid Assets - Ang et al. (2013)

This paper investigates the effect of illiquidity on the investment-consumption problem, where illiquidity is defined as the inability to freely trade an asset. The opportunity set consists of a riskless asset with return r , and two risky assets whose price processes follow a two-dimensional correlated GBM with drift parameters μ and ν , volatilities σ and ψ , and correlation ρ . The investor can trade freely in the riskless asset and the first risky asset, but the second risky asset can be only traded at stochastic times τ which are model by a Poisson process with intensity λ . The occurrences of the Poisson process represent randomly arriving trading opportunities when the investor can trade in the illiquid asset. The parameter λ is calibrated by noting that the expected time between trading opportunities implied by this model is $1/\lambda$. Alternatively, this can be thought of as the typical rebalancing turnover of the asset. The randomness of trading opportunities captures the non-marketability of certain assets as well as the waiting time before a counterparty is found for certain transactions.

The model assumes a CRRA investor with an infinite time horizon, i.e. optimization problem is given by:

$$\max_{C_t} \mathbb{E} \left[\int_0^\infty e^{-\beta t} U(C_t) dt \right] \quad (1)$$

$$\text{where } U(C) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 1 \\ \log(C) & \text{if } \gamma = 1 \end{cases} \quad (2)$$

The evolution of liquid and illiquid wealth are modeled separately and are given by:

$$\frac{dW_t}{W_t} = ((r + (\mu - r)\theta_t) - c_t) dt + \theta_t \sigma dZ_t^1 - \frac{dI_t}{W_t} \quad (3)$$

$$\frac{dX_t}{X_t} = \nu dt + \psi \rho dZ_t^1 + \psi \sqrt{1 - \rho^2} dZ_t^2 - \frac{dI_t}{X_t} \quad (4)$$

where θ is the fraction of liquid wealth allocated to the liquid risky asset, $c_t = C_t/W_t$ is the proportion of liquid wealth used for consumption, and dI is the amount transferred from liquid wealth to illiquid wealth when a trading opportunity arrives. So, the decision variables in this problem are θ, I, c and the value function is given by:

$$F(W_t, X_t) = \max_{\theta, I, c} \mathbb{E}_t \left[\int_t^\infty e^{-\beta(s-t)} U(C_s) ds \right] \quad (5)$$

By the homotheticity of the utility function, the value function can be written as:

$$F(W, X) = W^{1-\gamma} g\left(\frac{X}{W}\right), \quad \text{where } g(x) = F(1, x) \quad (6)$$

Furthermore, when a trading opportunity arrives the value function may discretely jump to a value F^* , so that the jump size is $F^* - F$ as the investor can rebalance the entirety of their portfolio. This can be written as:

$$F^*(W_t, X_t) = \max_{I \in [-X_t, W_t]} F(W_t - I, X_t + I) \quad (7)$$

Since F^* is also homothetic, there exists a function g^* such that $F^* = W^{1-\gamma} g^*\left(\frac{X}{W}\right)$. It is then argued, under the assumption of costless rebalancing, that we can rewrite the new value function as:

$$F^*(W_t, X_t) = GW_t^{1-\gamma} \left(1 + \frac{X_t}{W_t}\right)^{1-\gamma} \quad (8)$$

where G is a constant. The solution for the optimal decision variables is split into several parts (before the first rebalancing time, between subsequent rebalancing times, at rebalancing times). Assuming the investor begins with no allocation to the illiquid asset, the optimization problem is given by:

$$\max_{\theta, c} \mathbb{E}_0 \left[\int_0^\tau e^{-\beta t} \frac{1}{1-\gamma} C_t^{1-\gamma} dt + e^{-\beta \tau} F^*(W_\tau, X_\tau) \right] \quad (9)$$

The corresponding HJB equation in this case is:

$$0 = \max_{c, \theta} \left[-\beta F + \frac{1}{1-\gamma} (cW)^{1-\gamma} + F_W W (r + (\mu - r)\theta) - c \right. \\ \left. + \lambda(F^* - F) + \frac{1}{2} F_{WW} W^2 \theta^2 \sigma^2 \right] \quad (10)$$

Since $X = 0$, we have $F^* = GW^{1-\gamma} \left(1 + \frac{X}{W}\right)^{1-\gamma} = GW^{1-\gamma}$, and a verification argument gives the solution $F = K_0 W^{1-\gamma}$, where K_0 is a function of G and the model parameters.

The general HJB equation for the value function between rebalancing times is:

$$0 = \max_{c, \theta} \left[-\beta F + \frac{1}{1-\gamma} (cW)^{1-\gamma} + F_W W (r + (\mu - r)\theta) - c \right. \\ \left. + F_X X \nu + \lambda(F^* - F) + \frac{1}{2} F_{WW} W^2 \theta^2 \sigma^2 + F_{WX} W X \psi \sigma \rho \theta \right] \quad (11)$$

Substituting $F(W, X) = W^{1-\gamma} g\left(\frac{X}{W}\right)$ $F^* = GW^{1-\gamma} \left(1 + \frac{X}{W}\right)^{1-\gamma}$ and letting $x = \frac{X}{W}$ we can rewrite this as:

$$0 = \max_{c, \theta} \left[\frac{1}{1-\gamma} c^{1-\gamma} + \lambda G (1+x)^{1-\gamma} \right. \\ \left. + g(x) \left(-\beta - \lambda + (1-\gamma)(r + (\mu - r)\theta) - c \right) - \frac{1}{2} \gamma (1-\gamma) \sigma^2 \theta^2 \right) \\ \left. + g'(x) x (\nu - (r + (\mu - r)\theta) - c) - \gamma \psi \theta \rho \sigma + \gamma \theta^2 \sigma^2 \right. \\ \left. + g''(x) x^2 \left(\frac{1}{2} \sigma^2 \theta^2 + \frac{1}{2} \psi^2 - \psi \theta \rho \sigma \right) \right] \quad (12)$$

Finally, when a trading opportunity arrives, the investor chooses I_τ such that $\frac{X_\tau}{W_\tau} = x^*$. At x^* we impose the value matching and smooth pasting conditions $F(W, x^*W) = F^*(W, x^*W)$ and $F_W(W, x^*W) = F_X(W, x^*W)$. These conditions can be re-written as:

$$g(x^*) = G(1 + x^*)^{1-\gamma} \quad (13)$$

$$g'(x^*) = G(1 - \gamma)(1 + x^*)^{-\gamma} \quad (14)$$

These equations allows to solve for the investor's value function numerically, which in turn gives us the optimal portfolio before the first rebalancing opportunity:

$$\theta^* = -\frac{\mu - r}{\sigma^2} \frac{F_W}{F_{WW}W} \quad (15)$$

and between subsequent rebalancing opportunities:

$$\theta^* = -\frac{\mu - r}{\sigma^2} \frac{F_W}{F_{WW}W} - \frac{\psi\rho}{\sigma} \frac{F_{XW}X}{F_{WW}W} \quad (16)$$

Dusenberry's Ratcheting of Consumption: Optimal Dynamic Consumption and Investment Given Intolerance for any Decline in Standard of Living - Dybvig (1995)

This paper extends the Merton problem by imposing constraints on the optimal consumption policy. In particular, optimal consumption is forced to be a non-decreasing process so as to reflect what is referred to as ratcheting of consumption. This can also be viewed as an extreme version of habit formation in that consumption may remain constant for some time, but once the individual reaches a new level of consumption they grow accustomed to it and refuse to revert back to lower levels of consumption. An alternative interpretation to that of intolerance towards standard-of-living declines is to interpret this consumption pattern as pre-commitment to long-term expenditure.

Assume an infinitely-lived investor with utility function u and time-preference parameter δ is faced with a riskless asset with return rate r and a risky asset whose value is modeled with a GBM with parameters μ and σ . Starting with some endowment W_0 , they wish to choose optimal asset allocation and consumption decisions, subject to a starting consumption level $C_{0-} > 0$ and the constraint that their consumption never falls. The main stochastic control problem in this paper can then be formulated as follows:

$$\begin{aligned} & \max_{\alpha_t, c_t} \mathbb{E} \left[\int_0^\infty e^{-\delta t} u(c_t) dt \right] \\ \text{subject to: } & c_0 \geq C_{0-} \text{ and } c_t \geq c_s, \forall t > s \\ & w_t \geq 0 \text{ and satisfies } dw_t = w_t r dt + \alpha_t (\mu dt + \sigma dZ_t - r dt) - c_t dt \end{aligned} \quad (1)$$

where α_t is the investor's allocation to the risky asset and w_t is the the investor's wealth level at time t . Note that the problem is time-independent with c_{t-} and w_t as state variables. Furthermore, the utility function is assumed to be of the family of constant relative risk aversion (CRRA) utility functions, which are scale independent, which reduces the dimensionality of the state space to a single state variable w/c_{-} . So $u(c) = c^{1-R}/(1-R)$ for $R > 0, R \neq 1$ or $u(c) = \log(c)$ for $R = 1$, where R is the risk aversion parameter.

The solution is divided into several parts. The first step is to determine the feasibility region in which the optimization problem is guaranteed to be feasible. Based on the pricing of the consumption stream (with the properties described in the constraints of the control problem), it is shown that the problem is feasible if and only if $W_0 \geq C_{0-}/r$ and $r > 0$. This is not dissimilar from the analysis of the "danger zone" and "safe region" presented in Browne (1997). The intuition for this is to find the level of wealth needed to purchase a perpetual bond that can pay for future lifetime consumption. In this context, although increases in consumption are allowed they are also voluntary, and feasibility can be guaranteed by not

increasing the level of consumption when reaching this critical level of wealth. In short, if $C_{0-}/r > W_0$ the investor cannot generate the interest flow required to maintain consumption at its previous level.

The next part of the solution involves segmenting the feasible region into two subregions: $C_{0-}/r^* > W_0$ and $C_{0-}/r^* < W_0$, where r^* is the critical value which is the boundary of the region where consumption is maintained at its previous level. This critical value is given by $r^* = r \frac{R-R^*}{R}$, where $0 < R^* = \frac{\sqrt{(\delta+\kappa-r)^2+4r\kappa-(\delta+\kappa-r)}}{2r} < 1$ and $\kappa = \frac{(\mu-r)^2}{2\sigma^2}$. Note that this adds the additional feasibility condition $R > R^*$ to ensure that this critical value is positive. In the region $C_{0-}/r^* > W_0$, consumption is maintained at its previous level and is not a decision variable in the HJB equation:

$$0 = \max_{\alpha} \left\{ u(c) - \delta V + (wr + \alpha(\mu - r) - c)V_w + \frac{1}{2}V_{ww}\alpha^2\sigma^2 \right\} \quad (2)$$

Using the first order condition for optimal α and substituting this back into the HJB equation yields the PDE:

$$0 = u(c) - \delta V + (wr - c)V_w - \kappa \frac{(V_w)^2}{V_{ww}} \quad (3)$$

The boundary conditions associated with this PDE are given by the value function at $w = c_-/r$ computed to be $u(c)/\delta$ along with the smooth-pasting conditions to ensure the continuity of the partial derivatives of the value functions at the other boundary $w = c_-/r^*$. In the region where $C_{0-}/r^* < W_0$ it is argued that the optimal strategy is to immediately increase consumption to r^*w . Then the value function in this region satisfies $V(w, c_-) = V(w, r^*w)$.

Overall, the optimal consumption and asset allocation policies are given by:

$$\alpha_t = \frac{\mu - r}{R^*\sigma^2}(w - c/r) \quad (4)$$

$$c_t = \max \left\{ c_{0-}, k_1 \sup_{0 \leq s \leq t} \exp \left(\frac{1}{R} \left((r - \delta + \kappa) s + \frac{\mu - r}{\sigma} Z_s \right) \right) \right\} \quad (5)$$

$$\text{where } k_1 = \begin{cases} r^*W_0 & \text{when } r^*W_0 \geq c_{0-} \\ c_{0-} \left(1 + \frac{R}{R^*} \left(\frac{r^*W_0}{c_{0-}} - 1 \right) \right)^{R^*/R} & \text{otherwise} \end{cases} \quad (6)$$

The optimal solution involves a trade-off between achieving a smooth consumption pattern and profiting from market participation. It involves allocating a fixed amount to the riskless asset and a fixed proportion of wealth above the safe level to the risky asset. This is closely related to the constant proportion portfolio insurance problem of Black and Perold (1992)

and the survival and growth problems of Browne (1997). Note, however, that this proportion depends on market parameters and the impatience parameter, but not the risk aversion level.

Finally, this paper also considers several generalizations and extensions, namely:

- replacing the non-decreasing wealth constraint with a constraint that wealth cannot fall faster than a given rate;
- allowing for two different consumption goods of which one is constrained in the manner described above and the other is not
- introducing an upper bound on the risky stock allocation.

Minimizing the Probability of Ruin when Consumption is Ratcheted - Bayraktar and Young (2008)

This paper is related to Dybvig (1995) in that it deals with consumption patterns where consumption does not decrease (i.e. consumption is ratcheted). The two main differences is that this paper takes ratcheted consumption to be an exogenous process rather than a decision variable, and the region of interest is the “danger zone” which is the complement of the feasibility region described in Dybvig (1995). The reason for this is that they are interested in minimizing the probability of ruin, which is valid only when wealth is in the “danger zone” and there is a positive probability of ruin.

The opportunity set includes a riskless asset with return r and a GBM-driven risky asset with parameters μ and σ . The way the consumption process is modeled is to define it to be a function of maximum wealth. That is, given maximum wealth at time t , $M_t = \max \left[\sup_{0 \leq s \leq t} W_s, M_0 \right]$, the rate of consumption is assumed to be a positive, increasing function of maximum wealth, $c = c(M_t)$. Thus, wealth dynamics are given by:

$$dW_t = [rW_t + (\mu - r)\pi_t - c(M_t)] dt + \sigma\pi_t dB_t \quad (1)$$

where W_t is the investor’s wealth at time t , π_t is the risky asset allocation. Furthermore, the investor is assumed to have a random time of death, τ_d , which is exponentially distributed with parameter λ . If time to ruin is denoted by τ_0 , i.e. the first time that wealth equals 0, then minimum probability that the investor goes bankrupt before dying is given by:

$$\psi(w, m) = \inf_{\pi} \mathbb{P}[\tau_0 < \tau_d | W_0 = w, M_0 = m] = \inf_{\pi} \mathbb{E}_{w,m} [e^{-\lambda\tau}] \quad (2)$$

From this we can derive the HJB equation, which will be of the form:

$$0 = (rw + (\mu - r)\alpha - c(m)) \psi_w + \frac{1}{2} \sigma^2 \alpha^2 \psi_{ww} - \lambda \psi \quad (3)$$

along with the boundary conditions $\psi(w, m) = 0$ if $w \geq c(m)/r$ (safe region) and $\psi(w, m) = 1$ if $w \leq 0$ (bankruptcy).

The main verification theorem is then used to solve of the minimum probability of ruin ψ in different subregions: $w < c(m)/r \leq m$ and $m < c(m)/r$. The second of the two regions corresponds to the region where it is optimal to allow the maximum wealth M , and subsequently the level of consumption henceforth, to increase. In the first region the optimal risky allocation is given by:

$$\pi_t = \frac{\mu - r}{\sigma^2} \cdot \frac{1}{\gamma - 1} \left(\frac{c(M_t^\pi)}{r} - W_t^\pi \right) \quad (4)$$

where $\gamma = \frac{1}{2r} \left[(r + \lambda + \delta) + \sqrt{(r + \lambda + \delta)^2 - 4r\lambda} \right] > 1$ and $\delta = \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2$. The minimum probability of ruin is then given by:

$$\psi(w, m) = \left(1 - \frac{rw}{c(m)} \right)^\gamma \quad (5)$$

Notice that as wealth increases to the safe level $c(m)/r$, the allocation to the risk asset tends to 0, reflecting the decrease in the amount of risk that the investor needs to take in order to remain solvent. Furthermore, the probability of ruin increases with increased consumption and decreased wealth of riskless asset rate of return.

The solution to the problem in the second region is obtained by solving a related optimal controlled-stopper problem. However, the form of the solution is involved and is not mentioned here for brevity.

Shortfall Aversion - Guasoni et al. (2014)

Another attempt at incorporating habit formation is found in Guasoni et al. (2014). The main difference between this formulation and the one found in Dybvig (1995) is that the latter enforces a strict constraint on the optimal consumption process by not allowing it to decrease at any point in time. In this approach, this constraint is relaxed, but the distaste for declines in standard of living is not ignored. Instead it is modeled by assuming an asymmetric effect on utility due to increases versus decreases in consumption. That is, the utility function is adjusted to reflect that there is a greater level of disappointment attributable to spending cuts than pleasure gained by a similar increase in spending (relative to peak past spending). This allows the agent to decrease their consumption at any point in time if necessary, but this is penalized by a loss in overall utility.

Faced with a two-asset investment opportunity set (risky and riskless), the stochastic control problem is formulated as follows:

$$\max_{c,\pi} \mathbb{E} \left[\int_0^\infty \frac{(c_t/h_t^\alpha)^{1-\gamma}}{1-\gamma} dt \right] \quad (1)$$

Here, the utility function is a modification of the constant relative risk aversion family of utility functions. It depends on current spending c_t as well as a target level of spending h_t , which is taken to be the maximum past spending level: $h_t = \max \left[\bar{h}, \sup_{0 \leq s \leq t} c_s \right]$. The parameter $\gamma > 1$ represents the usual risk aversion parameter, while $0 < \alpha < 1$ is a new preference parameter that is associated with the level of shortfall aversion. The marginal utility of spending is higher for a cut in spending than an increase in spending and the ratio of these two at the target level is $1 - \alpha$. The utility function can be written as follows:

$$U(c, h) = \begin{cases} \frac{(ch^{-\alpha})^{1-\gamma}}{1-\gamma} & c \leq h \\ \frac{c^{(1-\alpha)(1-\gamma)}}{1-\gamma} & c > h \end{cases} \quad (2)$$

Another interpretation is that the risk aversion associated with a decrease in spending at h is γ , while the risk aversion associated an increase in spending above h is a α -weighted average of 1 and γ : $\gamma^* = \alpha + (1 - \alpha)\gamma < \gamma$. When there is no shortfall aversion, i.e. $\alpha = 0$, the solution is the one given in Merton (1969) for an investor with an infinite horizon and CRRA preferences where consumption and risky asset allocation are fixed proportions of wealth.

The full solution is not given here for brevity, but the process involves identifying two critical points: the “gloom” and “bliss” wealth-to-target ratios g and b . These are the points above which and below which spending cuts are necessary or spending should increase. The optimal solution is then stated in the three regions defined by these two points:

- Optimal risky allocation is equal to the Merton weight when the wealth-to-target ratio is below the gloom point. As wealth increases, the allocation to the risky asset also increases;
- Optimal consumption is equal to the target level when the wealth-to-target ratio is between the two critical points, and is proportional to wealth in each of the two other regions with weights $1/b$ and $1/g$ for the good and bad regions, respectively. Note that the quantity $1/g$ is equal to the proportion of wealth that gives optimal consumption in the Merton solution.

A few interesting properties to note include:

- The gloom point does not depend on shortfall aversion α ;
- The bliss point increases as shortfall aversion α increases;
- The gloom point is a fraction of the bliss point that depends mostly on the risk aversion γ and shortfall aversion α , but not very much on the market parameters;
- when $\alpha = 0$, the bliss and gloom points coincide and the problem becomes the original Merton problem; when $\alpha = 1$, the bliss point is infinity and the optimal solution involves no increases in consumption since the shortfall aversion level is extremely high.

Role of Index Bonds in an Optimal Dynamic Asset Allocation Model with Real Subsistence Consumption - Gong and Li (2006)

This paper investigates the role of real return bonds in the investment-consumption problem when it is included in the investment opportunity set while enforcing a lower bound on real consumption. In this context, the usual riskless asset (the nominal bond) becomes a risky asset as it is exposed to inflation risk and the new riskless asset (an index, or real return, bond) is available to the investor alongside the risky asset. The constraint on real consumption represents a form of habit formation in terms of real spending or pre-commitment.

The economy in this paper consists of a stochastic inflation rate, a stock, a nominal bond and a real return bond whose dynamics satisfy:

$$\frac{dP_t}{P_t} = \pi dt + \sigma_\pi dW_t \quad (1)$$

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dW_t \quad (2)$$

$$\frac{dB_t^N}{B_t^N} = (r_N - \pi) dt - \sigma_\pi dW_t \quad (3)$$

$$\frac{dB_t^R}{B_t^R} = r_R dt \quad (4)$$

where r_N and r_R are the nominal and real rates of interest, respectively, and W_t is a two dimensional Brownian motion which allows for inflation to be correlated with stock returns.

The optimal control problem is given by:

$$\begin{aligned} & \max_{c,w} \mathbb{E} \left[\int_0^\infty e^{-\rho t} u(c(t)) dt \right] \\ \text{subject to:} & \quad (1) \quad dX_t = [w_1(t)(\mu_S - r_R) + w_2(t)(r_N - \pi - r_R) \\ & \quad \quad \quad + X_t r_R - c(t)] dt + [w_1(t)\sigma_S - w_2(t)\sigma_\pi] dW_t \\ & \quad (2) \quad c(t) \geq \bar{c} \end{aligned} \quad (5)$$

where X denotes wealth level, w_1, w_2 is the wealth invested in the stock and nominal bond, respectively, c denotes *real* consumption, and \bar{c} is the lower bound on real consumption. Then the value function, $H(x)$, satisfies the HJB equation:

$$\begin{aligned} \sup_{c,w} \left\{ [w_1(\mu_S - r_R) + w_2(r_N - \pi - r_R) + x r_R - c] H_x \right. \\ \left. + u(c) + \frac{1}{2} \|w_1 \sigma_S - w_2 \sigma_\pi\|^2 H_{xx} \right\} = \rho H \end{aligned} \quad (6)$$

The utility function is assumed to be one of the CRRA family. In a manner similar to Dybvig (1995), the next steps involve finding two critical points: $\bar{x} = \frac{\bar{c}}{r_R}$ which is the boundary of the feasible region where the value function is equal to $H(\bar{x}) = \frac{u(\bar{c})}{\rho}$, and \hat{x} which is the solution to $H_x(\hat{x}) = u_c(\bar{c})$ which is the boundary of the region where optimal consumption according to the first order condition is below the minimum permissible consumption level.

The remainder of the paper involves solving for the value function by solving two PDEs:

$$H(x) = \begin{cases} H_1(x) & \bar{x} \leq x < \hat{x} \\ H_2(x) & x \geq \hat{x} \end{cases}$$

where H_2 is the solution to HJB equation after substituting in the first order conditions, and H_1 is the solution to the HJB equation after substituting the optimal asset allocation given by the FOC, but forcing the consumption to be the minimum level \bar{c} along with the boundary condition $H(\bar{c}/r_R) = \frac{u(\bar{c})}{\rho}$. Additionally, $H_x(\hat{x}) = u_c(\bar{c})$ is used in the solution as a smooth-pasting condition.

The results show that the optimal asset allocation strategy is once again of the class of portfolio insurance strategies. In general the optimal policy is to consume at the minimum allowed level and maintain a low level of risk at lower wealth levels, and to increase risk and consumption at higher wealth levels. Since both the stock and the nominal bond are considered risky assets in this framework, the allocation to these assets increases with wealth, while the allocation to the index bonds drops with wealth.

**Portfolio Selection with Subsistence Consumption Constraints and CARA Utility
- Shim and Shin (2014)**

Similar to Gong and Li (2006), this paper addresses the investment-consumption problem in the presence of a constraint on consumption level. Unlike Gong and Li (2006), however, inflation is not incorporated in the framework. The agent faces two assets, (riskless with rate of return r and risky following GBM dynamics with parameters μ and σ), and must determine optimal allocation and consumption policies so as to maximize their utility while ensuring that their consumption does not drop below a certain level:

$$\begin{aligned} & \max_{c,w} \mathbb{E} \left[\int_0^\infty e^{-\beta t} u(c_t) dt \right] \\ \text{subject to: } & (1) dX_t = [rX_t + \pi_t(\mu - r) - c_t] dt + \sigma\pi_t dB_t \\ & (2) c_t \geq R \end{aligned}$$

where X_t denotes wealth at time t , π_t denotes the allocation to the risky asset and R is the minimum consumption level which defines the subsistence consumption constraint. The utility function is assumed to be of CARA family, i.e. $u(c) = -\frac{e^{-\gamma c}}{\gamma}$. So the HJB equation is given by:

$$\sup_{c,\pi} \left[[rx + \pi(\mu - r) - c] V'(x) + \frac{1}{2} \sigma^2 \pi^2 V''(x) - \beta V(x) - \frac{e^{-\gamma c}}{\gamma} \right] = 0 \quad (1)$$

Like Gong and Li (2006), the initial endowment is assumed to be in the feasible region, i.e. $X_0 = x > R/r$, and there is a critical point $\bar{x} > R/r$, which satisfies $V'(\bar{x}) = u'(R) = e^{-\gamma R}$ at which the consumption constraint becomes binding. Based on the FOC the optimal consumption and allocation policies are given by:

$$c^* = \begin{cases} R, & \text{if } R/r < x \leq \bar{x} \\ -\frac{\log(V'(x))}{\gamma}, & \text{if } x \geq \bar{x} \end{cases} \quad (2)$$

$$\pi^* = -\frac{\theta V'(x)}{\sigma V''(x)} \quad (3)$$

These quantities are substituted back into the HJB equation to obtain 2 different PDEs in the two regions with boundary and smooth pasting conditions given by the quantities \bar{x} and R/r , from which the eventual solution is obtained.

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